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INVERSE LIMITS AND SMOOTHNESS OF CONTINUA

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Relations between the inverse limit operation and smoothness of continua are studied in this paper. Inverse systems with monotone bonding mappings are mainly discussed. It is shown that smoothness of continua is preserved under the inverse limits of inverse systems with monotone bonding mappings provided there exists a thread composed of points at which the factor spaces are smooth; in particular it follows that the inverse limit of an inverse sequence of smooth dendroids with monotone bonding mappings is a smooth dendroid if the corresponding thread does exist.

Topological spaces considered throughout this paper are assumed to be compact (thus Hausdorff, see [7], p. 165) and the mappings are assumed to be continuous. By a *continuum* we mean a compact connected space.

The following notation will be used. $\{X^\lambda, f^{\lambda\mu}, \Lambda\}$ denotes an inverse system of the topological spaces X^λ with continuous bonding mappings $f^{\lambda\mu}\colon X^\mu \to X^\lambda$ for any $\lambda \leqq \mu$, where $\lambda, \mu \in \Lambda$ and Λ is a set directed by the relation $\leqq$. We assume that $f^{\lambda\lambda}$ is the identity, and we denote by $X = \varprojlim \{X^\lambda, f^{\lambda\mu}, \Lambda\}$ the inverse limit space. Further, $\pi^\lambda\colon X \to X^\lambda$ denotes the projection from the inverse limit space into the λ-th factor space. In a particular case when Λ is the set N of natural numbers with the natural ordering $\leqq$ we write $\{X^i, f^i\}_{i=1}^\infty$ and π^i instead of $\{X^\lambda, f^{\lambda\mu}, \Lambda\}$ and π^λ respectively, where $f^i\colon X^{i+1} \to X^i$ are bonding maps, and then $\{X^i, f^i\}_{i=1}^\infty$ is called an inverse sequence.

Given a point $p \in X = \varprojlim \{X^\lambda, f^{\lambda\mu}, \Lambda\}$, we put $p^\lambda = \pi^\lambda(p) \in X^\lambda$ and we write $p = \{p^\lambda\}$. If $\Lambda = N$, we write $p = \{p^i\}$. Obviously we have

$$f^{\lambda\mu}(p^\mu) = p^\lambda \quad \text{for any} \quad \lambda, \mu \in \Lambda \quad \text{with} \quad \lambda \leqq \mu. \tag{1}$$

A point $p \in X$, i.e., a system of points $p^\lambda \in X^\lambda$ for $\lambda \in \Lambda$ satisfying (1) is called a *thread.*

Let a continuum X be given. Consider an arbitrary decomposition of X into two of its subcontinua A and B, i.e., $X = A \cup B$, and let $r(A, B)$ denote the number of components of $A \cap B$ less one. The *multicoherence degree* $r(X)$ is then defined by $r(X) = \sup \{r(A, B)\colon A \text{ and } B \text{ are subcontinua of } X \text{ and } A \cup B = X\}$ (see [6], p. 159: cf. [14], p. 83).

Let a point $p \in X$ be fixed. The *hereditarily multicoherence degree* $r(X, p)$ *of* X at the point p is defined by $r(X, p) = \sup \{r(C)\colon C \text{ is a subcontinuum of } X \text{ and } p \in C\}$.

In other words we have $r(X, p) = \sup \{r(A, B)\colon A \text{ and } B \text{ are subcontinua of } X \text{ and } p \in A \cap B\}$.

A continuum X is said to be *hereditary unicoherent at a point* $p \in X$ provided that the intersection of any two subcontinua of X, each of which contains p, is connected (see [8], p. 52). Thus a continuum X is hereditarily unicoherent at $p \in X$ if and only if $r(X, p)=0$.

We have the following

PROPOSITION 1. *Let* X *denote the inverse limit of an inverse system* $\{X^\lambda, f^{\lambda\mu}, \Lambda\}$ *of the continua* X^λ. *Let* $k \geqq 0$ *be a fixed integer. If there exists a thread* $p=\{p^\lambda\}$ *such that* $r(X^\lambda, p^\lambda) \leqq k$ *for each* $\lambda \in \Lambda$, *then* $r(X, p) \leqq k$.

PROOF. Note that X is a continuum ([3], 2.10, p. 236; cf. [7], 6.1.18, p. 436). Let A and B be subcontinua of X such that $p \in A \cap B$. Put $Y=A \cup B$, and $g^{\lambda\mu}=f^{\lambda\mu}|Y^\mu$, where $Y^\lambda=\pi^\lambda(Y) \subset X^\lambda$ for each $\lambda \in \Lambda$. Then $\{Y^\lambda, g^{\lambda\mu}, \Lambda\}$ is an inverse system with surjective mappings $g^{\lambda\mu}$, and Y is the inverse limit of this system (see [3], (2.8), p. 235; cf. [7], 2.5.7, p. 138). Further, we have $p^\lambda \in Y^\lambda$ for each $\lambda \in \Lambda$. Let $C^\lambda=\pi^\lambda(A) \cap \pi^\lambda(B) \subset Y^\lambda$ and put $h^{\lambda\mu}=g^{\lambda\mu}|C^\mu$. It follows from [3], (2.9), p. 235 that $\{C^\lambda, h^{\lambda\mu}, \Lambda\}$ is an inverse system having $A \cap B$ as its inverse limit. Since $\pi^\lambda(A)$ and $\pi^\lambda(B)$ are subcontinua of X^λ both containing p^λ, their intersection C^λ has no more than $k+1$ components by assumption. So by Lemma 1 of [11], p. 227, the intersection $A \cap B$ has no more than $k+1$ components. Thus the proof is complete.

COROLLARY 1. *Let* X *denote the inverse limit of an inverse system* $\{X^\lambda, f^{\lambda\mu}, \Lambda\}$ *of the continua* X^λ. *If there exists a thread* $p=\{p^\lambda\} \in X$ *such that* X^λ *is hereditarily unicoherent at* p^λ *for each* $\lambda \in \Lambda$, *then* X *is a continuum which is hereditarily unicoherent at the point* p.

The authors do not know whether the assumption that the points at which the continua X^λ are hereditarily unicoherent form a thread of the inverse system is essential in Corollary 1. Thus one can raise the following

PROBLEM 1. *Let* X *denote the inverse limit of an inverse system* $\{X^\lambda, f^{\lambda\mu}, \Lambda\}$ *of the continua* X^λ *each of which is hereditarily unicoherent at a point and such that all bonding mappings are onto. Does it follow that* X *is hereditarily unicoherent at some point?*

Recall that a continuum X is hereditarily unicoherent at a point $p \in X$ if and only if for every point $x \in X$ there exists in X a unique subcontinuum px which is irreducible between p and x (see [8], Theorem 1.3, p. 52). A continuum X is said to be *smooth at a point* $p \in X$ (in the sense of Gordh, [8], p. 52) if X is hereditarily unicoherent at p and for each convergent net $\{a_n; n \in D\}$ (where D is a directed set) of points of X the condition $\lim a_n=a$ implies that the net $\{pa_n; n \in D\}$ of subcontinua of X converges to the limit continuum pa. The point p is then called an *initial point* of X, and the set of all points at which a continuum X is smooth is called an *initial set* of X and is denoted by $I(X)$. If $I(X) \neq \emptyset$, then the continuum X is said to be *smooth*.

A *quasi-order* on a set X is a reflexive and transitive relation. A quasi-order on a topological space X is said to be *closed* if its graph is closed in $X \times X$. If a continuum X is hereditarily unicoherent at a point p, then the quasi-order $\leqq$ on X defined by the condition $x \leqq y$ if and only if $px \subset py$ is said to be a *weak cut-point order with respect to* p ([9], p. 63). It is known that a continuum X which is hereditarily unicoherent at a point $p \in X$ is smooth at this point if and only if the weak cutpoint order with respect to p is closed ([9], Theorem 3.1, p. 65).

We discuss now some relations between smoothness of continua and the inverse limit operation. We start with an example.

Let X be the simplest Knaster indecomposable continuum with one end point (see [10], § 48, V, Example 1, p. 204 and Fig. 4, p. 205). It is well known that X is the inverse limit of an inverse sequence $\{X^i, f^i\}_{i=1}^{\infty}$, where X^i is the closed unit segment $[0, 1]$ and $f^i\colon [0, 1] \to [0, 1]$ is a fixed mapping defined by $f^i(t) = 1 - |2t-1|$, $t \in [0, 1]$, for each $i = 1, 2, \dots$. Note that each continuum X^i is hereditarily unicoherent at each point and it is smooth at each point, each bonding mapping is open, and yet the limit continuum X is hereditarily unicoherent at each point while it is smooth at none. So we see that smoothness of continua is not preserved by the inverse limit operation, even if the inverse limit space is hereditarily unicoherent at each point and if bonding mappings are very simple ones (in particular open). Therefore a natural question arises concerning conditions (on factor spaces and/or on bonding mappings) under which the inverse limit continuum is smooth at a point provided that the factor spaces are so. The following theorem gives a partial answer to this question.

THEOREM 1. *Let X be the inverse limit of an inverse system $\{X^\lambda, f^{\lambda\mu}, \Lambda\}$, where X^λ are continua and $f^{\lambda\mu}$ are monotone mappings. If there exists a thread $p = \{p^\lambda\}$ such that X^λ is smooth at p^λ for each $\lambda \in \Lambda$, then X is a continuum which is smooth at the point p.*

PROOF. By Corollary 1 the inverse limit space X is a continuum which is hereditarily unicoherent at p. By Theorem 3.1 of [9], p. 65 for each $\lambda \in \Lambda$ there exists a closed weak cutpoint order $\leqq_\lambda$ with respect to the point p^λ. Define a relation $\leqq$ on X by $x \leqq y$ if and only if $x^\lambda \leqq_\lambda y^\lambda$ for each $\lambda \in \Lambda$. Note that $\leqq$ is transitive and reflexive, i.e., it is a quasi-order. We claim that $\leqq$ is closed. To see this, consider two convergent nets $\{x_n; n \in D\}$ and $\{y_n; n \in D\}$ of points of X having points x and y of X as their limits respectively. Assume that $x_n \leqq y_n$ for each $n \in D$. Thus $x_n^\lambda = \pi^\lambda(x_n) \leqq_\lambda y_n^\lambda = \pi^\lambda(y_n)$ for each $\lambda \in \Lambda$ by the definition of the quasi-order $\leqq$ on X. Since a net $\{x_n; n \in D\}$ of points in the inverse limit space X converges to a limit point x if and only if the nets $\{x_n^\lambda; n \in D\}$ converge to x^λ for each $\lambda \in \Lambda$ (see [7], 2.3.34, p. 119), and since each quasi-order $\leqq_\lambda$ is closed, we have $x^\lambda \leqq_\lambda y^\lambda$ for each $\lambda \in \Lambda$, and thus the claim is proved.

To complete the proof we need only to show that the quasi-order defined above is just the weak cutpoint order with respect to p, i.e., that $x \leqq y$ holds if and only if $px \subset py$. To this end let us take for $\lambda \leqq \mu$ the partial mapping $g^{\lambda\mu} = f^{\lambda\mu}|p^\mu y^\mu$ from the unique irreducible continuum $p^\mu y^\mu$ into the continuum X^λ.

It follows from Theorem 4.1 (ii) of [8], p. 56 that $g^{\lambda\mu}(p^\mu y^\mu) = p^\lambda y^\lambda$, so we can consider the mapping $g^{\lambda\mu}\colon p^\mu y^\mu \to p^\lambda y^\lambda$ as a surjection. Obviously we have $g^{\lambda\mu}(p^\mu) = p^\lambda$ and $g^{\lambda\mu}(y^\mu) = y^\lambda$. Since $\{\pi^\lambda(X), f^{\lambda\mu}|\pi^\mu(X), \Lambda\}$ is an inverse system ([3], 2.8, p. 235), it can be easily verified that $\{p^\lambda y^\lambda, g^{\lambda\mu}, \Lambda\}$ is also an inverse system. Let L denote its inverse limit. Obviously $p, y \in L$. Further, L is a continuum which is irreducible from p to y. Indeed, if M is a subcontinuum of L containing the points p and y, then for each $\lambda \in \Lambda$ the set $\pi^\lambda(M)$ is a continuum containing p^λ and y^λ and contained in $p^\lambda y^\lambda$. Therefore $\pi^\lambda(M) = p^\lambda y^\lambda$ by irreducibility of $p^\lambda y^\lambda$, whence by [3], 2.8, p. 235 it follows that $M = \varprojlim \{p^\lambda y^\lambda, g^{\lambda\mu}, \Lambda\} = L$. Thus $L = py$, since by hereditary unicoherence of X at p there is only one continuum irreducible from p to y ([8], Theorem 1.3, p. 52).

Furthermore, since $\{p^\lambda y^\lambda, g^{\lambda\mu}, \Lambda\}$ is an inverse system, we conclude that the condition $x \leqq y$, i.e., $x^\lambda \leqq_\lambda y^\lambda$ for each $\lambda \in \Lambda$ (which means that $p^\lambda x^\lambda \subset p^\lambda y^\lambda$ for each $\lambda \in \Lambda$), is equivalent to $px \subset py$. So the proof is finished.

The remainder of the paper deals with metrizable spaces only, therefore we restrict the inverse systems to be considered as inverse sequences (cf. [7], Corollary 4.2.4, p. 324 for justification). We recall that the inverse limit of an inverse sequence of metrizable spaces is metrizable ([7], 4.2.5, p. 325) and that the inverse limit of an inverse system of continua (i.e. of compact connected topological spaces) is a continuum ([7], 6.1.18, p. 436), whence it follows that

(2) the inverse limit of an inverse sequence of metric continua is a metric continuum.

By a *dendroid* we mean a metric continuum which is arcwise connected and hereditarily unicoherent (i.e. an arcwise connected metric continuum such that the intersection of each two of its subcontinua is connected). Nadler ([11], Theorem 4, p. 229) and Bellamy ([2], Lemma 1, p. 192) have proved the following result about inverse limits of dendroids:

THEOREM A (Nadler, Bellamy). *Let X denote the inverse limit of an inverse sequence $\{X^i, f^i\}_{i=1}^{\infty}$ where X^i is a dendroid for each $i=1, 2, \ldots$.*

1. *If X is arcwise connected, then X is a dendroid.*

2. *If X is locally connected, then X is a dendrite.*

3. *If X^i is a dendrite and $f^i\colon X^{i+1} \to X^i$ is monotone for each $i=1, 2, \ldots$, then X is a dendrite.*

4. *If the mapping $f^i\colon X^{i+1} \to X^i$ is monotone for each $i=1, 2, \ldots$, then X is a dendroid.*

Recall that for dendroids the notion of smoothness presented above and due to G. R. Gordh, Jr. ([8], p. 52) coincides with a previous one due to the first author and C. A. Eberhart ([5], p. 298). Thus Theorems 1 and A (Part 4) imply the following

COROLLARY 2. *Let X be the inverse limit of an inverse sequence $\{X^i, f^i\}_{i=1}^{\infty}$ of the dendroids X^i with monotone mappings $f^i\colon X^{i+1} \to X^i$. If there exists a thread $p = \{p^i\}$ such that X^i is smooth at p^i for each $i=1, 2, \ldots$, then X is a dendroid which is smooth at p.*

Similarly to a previous question (see Problem 1) the authors do not know whether the existence of a thread composed of initial points of X^i (i.e. of points at which each X^i is smooth) is an essential assumption in Corollary 2. Thus we have

PROBLEM 2. *Let a dendroid X be the inverse limit of an inverse sequence of smooth dendroids with monotone bonding mappings. Is then X smooth?*

Monotonicity of bonding mappings is an essential assumption in Corollary 2 even in the case when the limit continuum is a dendroid. This can be seen by the following

EXAMPLE. There exists an inverse sequence $\{X^i, f^i\}_{i=1}^{\infty}$ such that X^i is a finite dendrite, $X^i \subset X^{i+1}$, and f^i is a retraction for each $i=1, 2, \ldots$ whose inverse limit space X is a non-smooth (and even non-contractible and non-selectible) dendroid.

Indeed, let A_n be the line segment joining $(-1,0)$ and $(1,2^{-n})$ in the plane, for $n=0,1,2,\ldots$ and let T be the line segment joining $(-1,0)$ and $(1,0)$. Then $D_1=T\cup\bigcup_{n=0}^{\infty}A_n$ is a dendroid (the so called harmonic fan). Let D_2 be the reflection of D_1 about the origin, and put $X=D_1\cup D_2$ (see [12], Fig. 1, p. 372). It is evident that X is a non-smooth dendroid, and it is known that X is non-contractible ([13], Theorem 2.1, p. 838) and that the hyperspace $C(X)$ of its subcontinua admits no continuous selection ([12], Theorem 2, p. 372). For each $i=1,2,\ldots$ put $D_1(i)=$ $=T\cup\bigcup_{n=1}^{i}A_n$, let $D_2(i)$ denote the reflection of $D_1(i)$ about the origin, and define $X^i=D_1(i)\cup D_2(i)$. Thus $X^i\subset X$ is a finite dendrite for each $i=1,2,\ldots$ and we have

$$T\subset X^1\subset X^2\subset\ldots\subset X^i\subset X^{i+1}\subset\ldots\subset X=\overline{\bigcup_{i=1}^{\infty}X^i}.$$

We define the mapping $f^i\colon X^{i+1}\to X^i$ as follows:

$$f^i(x,y)=\begin{cases}(x,y) & \text{if } (x,y)\in X^i,\\ (x,0) & \text{if } (x,y)\in X^{i+1}\setminus X^i.\end{cases}$$

Thus f^i projects the $(i+1)$-th line segment A_{i+1} in $D_1(i+1)$ and its reflection about the origin in $D_2(i+1)$ perpendicularly onto T, and f^i is the identity on the rest. So $f^i\colon X^{i+1}\to X^i$ is a retraction. The equality $X=\varprojlim\{X^i,f^i\}_{i=1}^{\infty}$ can be seen by Theorem I of [1], p. 348.

In light of the above example one can ask questions about conditions (regarding the bonding mappings f^i) under which some other properties such as contractibility or the existence of a continuous selection on the hyperspace of subcontinua are transferred from X^i to $X=\varprojlim(X^i,f^i)$. Studying these and similar problems is left for the future.

Let us keep our attention for a while on a very particular kind of dendroids, nemely on fans. By a *fan* we understand a dendroid having exactly one ramification point (called the *top* of the fan).

Observe the following easy characterization of fans.

PROPOSITION 2. *A dendroid F is a fan with top p if and only if for every two points $x,y\in F$ the condition $px\cap py\setminus\{p\}\neq\emptyset$ implies either $px\subset py$ or $py\subset px$ (we consider here an arc as a degenerate fan whose top is an arbitrary point).*

THEOREM 2. *Let X denote the inverse limit of an inverse sequence $\{X^i,f^i\}_{i=1}^{\infty}$, where X^i is a fan and f^i is a monotone mapping of X^{i+1} into X for each $i=1,2,\ldots$. Then X is a fan (an arc or a singleton).*

PROOF. Let p_i be the top of the fan X^i for each $i=1,2,\ldots$. Applying Theorem 12 of [4], p. 32 we have $f^i(p_{i+1})=p_i$ so that there exists a thread $p\in X$ whose coordinates $\pi^i(p)=p^i=p_i$ are just the tops of the fans X^i. It follows from Part 4 of Theorem A that X is a dendroid. We will show that X is a fan with top p. To this end, let x and y be two distinct points of X such that $px\cap py\setminus\{p\}\neq\emptyset$. Let $q\in px\cap$

$\cap py \setminus \{p\}$. Consider the partial mappings $g^i = f^i | p^{i+1}x^{i+1}$ and $h^i = f^i | p^{i+1}y^{i+1}$ of $p^{i+1}x^{i+1}$ and $p^{i+1}y^{i+1}$ into X^i for each $i = 1, 2, \dots$. They are monotone (see e.g. [5], Proposition 1, p. 307), whence it follows that $g^i(p^{i+1}x^{i+1})$ is the arc $p^i x^i$ and, similarly, $h^i(p^{i+1}y^{i+1})$ is the arc $p^i y^i$ (cf. [14], Chapter IX, (1.1), p. 165). Thus we can consider the mappings $g^i\colon p^{i+1}x^{i+1} \to p^i x^i$ and $h^i\colon p^{i+1}y^{i+1} \to p^i y^i$ as monotone and onto for each $i = 1, 2, \dots$. They preserve end points of the arcs, whence we conclude that $\{g^i, p^i x^i\}_{i=1}^{\infty}$ and $\{h^i, p^i y^i\}_{i=1}^{\infty}$ are inverse sequences, the inverse limits of which are the arcs px and py, respectively (see [3], Theorem 4.8, p. 244). Thus we have $q^i \in p^i x^i \cap p^i y^i \setminus \{p^i\}$ for each $i = 1, 2, \dots$. Since the X^i are fans, Proposition 2 implies that for every $i = 1, 2, \dots$ we have either $p^i x^i \subset p^i y^i$ or $p^i y^i \subset p^i x^i$. Note that since g^i and h^i are monotone and onto, either the former inclusion holds for all $i = 1, 2, \dots$ or the latter one is satisfied for all $i = 1, 2, \dots$. In the first case we have $px \subset py$, in the second one the inclusion $py \subset px$ is true; therefore X is a fan with top p by Proposition 2.

The next corollary summarises some results on inverse sequences of dendroids with monotone bonding mappings.

COROLLARY 3. *The following properties are preserved under the inverse limits of inverse sequences with monotone bonding mappings:* (a) *to be an arc;* (b) *to be a dendrite;* (c) *to be a dendroid;* (d) *to be a smooth dendroid, provided that there is a thread of initial points;* (e) *to be a fan;* (f) *to be a smooth fan.*

PROOF. (a) see [3], Theorem 4.8, p. 244; (b) [11], Part 3 of Theorem 4, p. 229; (c) [2], Lemma 1, p. 192; (d) Corollary 2; (e) Theorem 2; (f) see (d) and (e) above and note that the existence of a proper thread is shown in the proof of (e).

Results of the present paper will be applied by the first author in a forthcoming paper to the construction of a dendroid composed of endpoints and of ramification points only.

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