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Property of Kelley for confluent retractable continua

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Abstract

The property of Kelley for confluent retractable continua is studied. It is shown that a confluent retractable continuum has the property of Kelley if and only if each of its proper subcontinua has the property. An example is constructed of a confluent retractable continuum without the property of Kelley. © 2001 Elsevier Science B.V. All rights reserved.

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A *continuum* means a compact connected metric space, and a *mapping* means a continuous function.

A metric continuum X is said to have the *property of Kelley* provided that for each point $x \in X$, for each subcontinuum K of X containing x and for each sequence of points x_n converging to x there exists a sequence of subcontinua K_n of X containing x_n and converging to the continuum K (see, e.g., [9, Definition 16.10, p. 538]). The property, introduced by Kelley as Property 3.2 in [6, p. 26], has been used to study hyperspaces, in particular their contractibility (see, e.g., Chapter 16 of [9], where references for further results in this area are given). Now the property, which has been recognized as an important tool in investigation of various properties of continua, is interesting in its own right, and has numerous applications to continuum theory. Many of them are not related to hyperspaces.

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Given a continuum X with a metric d, we let 2^X denote the hyperspace of all nonempty closed subsets of X equipped with the Hausdorff metric H defined by

$$H(A, B) = \max\bigl\{\sup\{d(a, B)\colon a \in A\}, \sup\{d(b, A)\colon b \in B\}\bigr\}$$

(see, e.g., [9, (0.1), p. 1 and (0.12), p. 10]). Further, we denote by $C(X)$ the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X, and, for a point $p \in X$, we denote by $C(p, X)$ the family of all subcontinua of X containing the point p.

The reader is referred to Nadler's book [9] for needed information on the structure and properties of hyperspaces.

A mapping $f : X \to Y$ between continua X and Y is said to be:

- *confluent* provided that for each subcontinuum Q of Y and for each component C of $f^{-1}(Q)$ we have $f(C) = Q$;
- a *retraction* provided that $Y \subset X$ and $r|Y$ is the identity.

A subcontinuum Y of a continuum X is called a *retract* of X provided that there exists a retraction from X onto Y. A continuum X is said to be *retractable* provided that each subcontinuum of X is a retract of Y.

Let $\mathfrak{M}$ be a class of mappings between continua. If, for each subcontinuum Y of a continuum X there exists a retraction $r : X \to Y$ such that $r \in \mathfrak{M}$, then X is said to be $\mathfrak{M}$ *retractable*. Confluent and open retractable continua were investigated by the third named author in [5]. In particular, it is showed there that the dyadic solenoid is open retractable [5, Example]. The proof, however, shows even more: if X is either a solenoid or a Knaster-type continuum (i.e., the inverse limit of arcs with open bonding mappings), then X is open retractable. In both cases every proper subcontinuum of X is an arc. Thus one can ask if every continuum all proper subcontinua of which are arcs is open retractable. We show that this is not the case, proving that such continua must have the property of Kelley. The problem of a characterization of open or confluent retractable continua with all subcontinua being arcs remains open.

Observation 1. *A continuum X has the property of Kelley if and only if the following condition holds.*

(1.1) *For each $\varepsilon > 0$, for each $K \in C(X)$, for each point $x_0 \in K$ and for each sequence of points x_n converging to x_0 there exists a sequence of continua $K_n \in C(x_n, X)$ such that if a subsequence K_{n_j} is convergent, then $H(\operatorname{Lim}_j K_{n_j}, K) < \varepsilon$.*

The main result of the paper is the following theorem.

Theorem 2. *If a continuum X is confluent retractable, and if each proper subcontinuum of X has the property of Kelley, then the whole continuum X also has the property of Kelley.*

Proof. Let $\varepsilon > 0$, a proper subcontinuum K of X, a point $x_0 \in K$ and a sequence of points x_n converging to x_0 be given as in condition (1.1). We will construct the needed sequence of continua $K_n \in C(x_n, X)$.

To this aim define

$$L=\bigcup\bigl\{P\in C(X)\colon\ P\cap K\neq\emptyset \text{ and } H(P,K)\leqslant\varepsilon/2\bigr\},$$

and observe that L is a continuum. Obviously we may choose ε in such a way that $L\neq X$. Let $r:X\to L$ be a confluent retraction. Then the sequence of points $r(x_n)$ tends to the point $r(x_0)=x_0\in K\subset L$. Since L has the property of Kelley, there is a sequence of continua $P_n\in C(r(x_n),L)$ converging to K. Let K_n be the component of $r^{-1}(P_n)$ containing the point x_n. Since r is confluent, $r(K_n)=P_n$.

We have to show that if a subsequence $\{K_{n_j}\colon\ j\in\mathbb{N}\}$ is convergent, then $H(\operatorname{Lim}_j K_{n_j}, K)<\varepsilon$. So, suppose on the contrary that this implication does not hold. Then there is a sequence $\{K_{n_j}\colon\ j\in\mathbb{N}\}$ that converges to a continuum K' and that

$$H(K,K')\geqslant\varepsilon. \tag{2.1}$$

Thus the sequence $r(K_{n_j})$ has $r(K')\subset L$ as its limit. But $r(K_{n_j})=P_{n_j}$ for each $j\in\mathbb{N}$ by confluence of r, and this sequence tends to K. Consequently, $r(K')=K$.

Further, since for each $j\in\mathbb{N}$ we have $x_{n_j}\in K_{n_j}$ and since the points x_{n_j} tend to x_0, and the continua K_{n_j} tend to the continuum K', we get $x_0\in K'$. Therefore $x_0\in K\cap K'\neq\emptyset$, whence we see that $K\cup K'$ is a continuum.

If $H(K,K\cup K')\leqslant\varepsilon/2$, then $K\cup K'\subset L$ by the definition of L. However $r|L$ is the identity, thus $r|K'$ is the identity, whence $K'=K$, contrary to (2.1). Consequently $H(K,K\cup K')>\varepsilon/2$. Consider an order arc from the singleton $\{x_0\}$ to the continuum K'. Let K'' be the first element of this order arc satisfying

$$H(K,K\cup K'')=\varepsilon/2. \tag{2.2}$$

Then $K''\subset K'\cap L$ and $K\subsetneq K\cup K''$. The former inclusion implies that $r|K''$ is the identity, and the latter one leads to $K''\setminus K\neq\emptyset$. But the inclusion $K''\subset K'$ gives $r(K'')\subset r(K')=K$, and since $r|K''$ is the identity, we have $K''\subset K$, contrary to (2.2). The proof is complete. □

Corollary 3. *If a continuum is confluent retractable, then it has the property of Kelley if and only if each of its proper subcontinua has the property.*

Proof. One implication is Theorem 2. The other one follows since retractions as well as confluent mappings preserve the property of Kelley [12, Theorem 2.9, p. 294, and Theorem 4.3, p. 296]. □

Corollary 4. *If a continuum X is confluent retractable, and if each proper subcontinuum of X is locally connected, then X has the property of Kelley.*

Corollary 5. *If a continuum X is confluent retractable, and if each proper subcontinuum of X is an arc, then X has the property of Kelley.*

Remark 6. A wider class than confluent mappings is the class of weakly confluent ones. Recall that a mapping $f:X\to Y$ between continua X and Y is said to be *weakly confluent* provided that for each subcontinuum Q of Y there is a component C of $f^{-1}(Q)$ such that $f(C) = Q$. The assumption that the retraction is confluent neither can be deleted from Theorem 2 nor can be relaxed to being weakly confluent. This is shown in an example below.

Example 7. There is a weakly confluent retractable continuum X such that each proper subcontinuum of X is an arc and X does not have the property of Kelley.

Proof. The needed continuum X can be obtained, for example, as a modification of the simplest Knaster indecomposable (i.e., buckethandle) continuum D. We will use the description of D as in [7, §48, V, Example 1 and Fig. 4, pp. 204 and 205].

To describe the modification, we represent the standard Cantor ternary set $\mathcal{C}$ of reals in the closed unit interval $[0,1]$ as the union

$$\mathcal{C} = \{0\} \cup \bigcup \{\mathcal{C}_n \colon n \in \mathbb{N}\},$$

where, for each positive integer n, the set

$$\mathcal{C}_n = \mathcal{C} \cap \left(\left[\frac{2}{3^n}, \frac{1}{3^{n-1}}\right]\right)$$

is a copy of $\mathcal{C}$.

Replace the end point $(0,0)$ of D by the straight line segment $\{0\} \times [-1,0]$, and, for each odd n, replace the nth portion $\mathcal{C}_n \times \{0\} \subset D$ of the Cantor set $\mathcal{C} \times \{0\} \subset D$ by the Cantor bundle of the straight line segments of the form $\mathcal{C}_n \times [-1,0]$. Therefore the resulting continuum X is the union of:

1. all semicircles with nonnegative ordinates of points, with center $(\frac{1}{2}, 0)$, passing through every point of the Cantor set $\mathcal{C} \times \{0\}$;
2. all semicircles with nonpositive ordinates of points, which have for each *even n* the center at $(\frac{5}{2\cdot 3^n}, 0)$ and pass through each point of the set $\mathcal{C}_n \times \{0\}$;
3. the straight line segment $\{0\} \times [-1,0]$;
4. all straight line segments $\{x\} \times [-1,0]$, where $x \in \mathcal{C}_n$ for *odd n*;
5. all semicircles with ordinates of points $\leqslant -1$ which have for each *odd n* the center at $(\frac{5}{2\cdot 3^n}, -1)$ and pass through each point of the set $\mathcal{C}_n \times \{-1\}$.

It is evident from the construction that X is an indecomposable continuum without the property of Kelley, each of whose proper subcontinua is an arc. Then X is retractable (since the arc is an absolute retract), the retraction is weakly confluent (since each mapping onto an arc is weakly confluent [10, Lemma, p. 236]), and each of its subcontinua has the property of Kelley, while X does not have the property.

Remark 8. The assumption of retractability of the continuum X onto each of its proper subcontinua is also indispensable in Theorem 2, in the sense that it is not enough to assume that each of proper subcontinua of a continuum X has the property of Kelley and is a confluent image (in place of being a confluent retract) of X to obtain the conclusion of the theorem. This can be seen again by the same Example 7. Namely shrinking the segment $\{0\} \times [-1,0] \subset X$ to the point $(0,0)$ we get a monotone mapping from X back onto the buckethandle continuum D; next D can be mapped onto an arc under an open mapping. Since the composition of monotone and open mappings is confluent, each arc is a confluent image of X. It is also a retract of X. But the retraction cannot be confluent, according to Theorem 2, because X does not have the property of Kelley.

According to Corollary 3, having the property of Kelley for confluent retractable continua is equivalent to having the property hereditarily, i.e., that every of the subcontinua of the whole continuum also has this property. Such continua were investigated in [1]. One may wonder if this property implies confluent retractability. We show that this is not the case.

Example 9. There exists a continuum X having the property of Kelley hereditarily which is not (confluent) retractable.

Proof. Let P be the pseudo-arc, and let H be half line approximating P, i.e., $\operatorname{cl} H \setminus H = P$. Define $X = H \cup P$. Then every subcontinuum of X is homeomorphic either to X, or to the arc, or to P. Since the arc and the pseudo-arc have the property of Kelley, [9, (16.11), p. 538, and (16.26), p. 552], to prove that X has the property of Kelley hereditarily it is enough to show that X has the property of Kelley.

Let $K \in C(X)$, a point $x \in K$ and a sequence of points x_n tending to x be given. The only nontrivial case is when $K \subset P$ and $x_n \in H$. Let $\mu : C(X) \to [0,1]$ be any Whitney map (see [9, (0.50), p. 24] for the definition), and let K_n be any continuum in $C(x_n, X)$ satisfying $\mu(K_n) = \mu(K)$ for each $n \in \mathbb{N}$. Then for any convergent subsequence $\{K_{n_i}\colon i \in \mathbb{N}\}$ of the sequence $\{K_n\colon n \in \mathbb{N}\}$ we have $x \in \operatorname{Lim}_i K_{n_i}$ and $\mu(\operatorname{Lim}_i K_{n_i}) = \mu(K)$. Since P is hereditarily indecomposable, the two conditions imply that $\operatorname{Lim}_i K_{n_i} = K$, and, consequently, $\operatorname{Lim} K_n = K$. This shows that X has the property of Kelley hereditarily.

Since there is no mapping of X onto P, the continuum X is not retractable. The proof is then complete. □

The assumption that each proper subcontinuum of X has the property of Kelley is essential in Theorem 2. The next example shows this.

Example 10. There is a confluent retractable continuum without the property of Kelley.

Proof. Let P_1 and P_2 be two copies of the pseudo-arc such that $P_1 \cap P_2 = \{p\}$ for some point p, and let $X = P_1 \cup P_2$. We will show that X is the needed continuum.

It is easy to verify that X does not have the property of Kelley. To show that it is confluent retractable take a nondegenerate subcontinuum M of X and consider three cases.

Case 1. $M \subset P_1$. Since the pseudo-arc is homogeneous, see [2, Theorem 13, p. 740], there exists a homeomorphism $h : P_2 \to P_1$ such that $h(p) = p$. Since the pseudo-arc is retractable [3], there is a retraction $r_1 : P_1 \to M$. Define a mapping $r : X \to M$ by

$$r(x) = \begin{cases} r_1(x), & \text{if } x \in P_1, \\ r_1(h(x)), & \text{if } x \in P_2. \end{cases}$$

Then r is a retraction. Since M is homeomorphic to the pseudo-arc [8], which is known to be hereditarily indecomposable [2, Theorem 10, p. 737] and since each mapping onto any hereditarily indecomposable continuum is confluent [4, Theorem 4, p. 243], we infer that r is confluent.

Case 2. $M \subset P_2$. This case is similar to the previous one.

Case 3. M is contained neither in P_1 nor in P_2. Since p is a cut point of X, it is also a cut point of M. Then the intersections $M_1 = M \cap P_1$ and $M_2 = M \cap P_2$ are subcontinua of X. For each $i \in \{1, 2\}$ choose a point $q_i \in M_i$ such that the composant K_i of M_i which contains q_i does not contain the point p. By Theorem 1 of [11, p. 131] there exist retractions $r_i : P_i \to M_i$ such that $r_i(P_i \setminus M_i) \subset K_i$. Define a mapping $r : X \to M$ by

$$r(x) = \begin{cases} r_1(x), & \text{if } x \in P_1, \\ r_2(x), & \text{if } x \in P_2. \end{cases}$$

Then r is a retraction. To show that it is confluent, let Q be a subcontinuum of M and let C be a component of $f^{-1}(Q)$. We consider three subcases.

(a) $Q \subset M_1$. Then, since $r^{-1}(p) = p$, it follows that $r^{-1}(Q) = r_1^{-1}(Q)$. Further, since M_1 is hereditarily indecomposable, r_1 is confluent by the previously quoted argument. Thus $Q = r_1(C) = r(C)$.

(b) $Q \subset M_2$. This subcase is similar to the previous one.

(c) Q is contained neither in P_1 nor in P_2. Then $p \in Q$. Note that the intersections $Q_1 = Q \cap P_1$ and $Q_2 = Q \cap P_2$ are subcontinua of X.

If $Q_1 \neq M_1$ and $Q_2 \neq M_2$, then $Q_1 \cap K_1 = \emptyset$ and $Q_1 \cap M_2 = \emptyset$. Thus $r^{-1}(Q) = Q$, whence $C = Q$ and $r(C) = Q$, as needed.

If $Q_1 \neq M_1$ and $Q_2 = M_2$, then $r^{-1}(Q) = Q_1 \cup P_2$. Thus $C = Q_1 \cup P_2$, and again $r(C) = Q$.

If $Q_1 = M_1$ and $Q_2 \neq M_2$, then $C = Q_2 \cap P_1$, and $r(C) = Q$.

Finally if $Q_1 = M_1$ and $Q_2 = M_2$, then $r^{-1}(Q) = X$. Thus $C = X$, and $r(C) = Q$.

Therefore we have shown that the retraction r is confluent. The proof is complete. □

The construction used in Example 10 leads to the following question that is of some interest.

Question 11. Let a continuum be confluent retractable and hereditarily decomposable. Must it have the property of Kelley?

In the light of Example 9 the next question is interesting.

Question 12. Let X be a continuum with the property of Kelley such that each proper subcontinuum of X is an arc. Is then X open (or confluent) retractable?

References

[1] G. Acosta, A. Illanes, Continua which have the property of Kelley hereditarily, Topology Appl. 102 (2000) 151–162.

[2] R.H. Bing, A homogeneous indecomposable plane continuum, Duke Math. J. 15 (1948) 729–742.
[3] J.L. Cornette, Retracts of the pseudo-arc, Colloq. Math. 19 (1968) 235–239.
[4] H. Cook, Continua which admit only the identity mapping onto non-degenerate subcontinua, Fund. Math. 60 (1967) 241–249.
[5] A. Illanes, Confluent and open retractable continua, Acta Math. Hungar. 84 (1999) 1–8.
[6] J.L. Kelley, Hyperspaces of a continuum, Trans. Amer. Math. Soc. 52 (1942) 22–36.
[7] K. Kuratowski, Topology, Vol. 2, Academic Press, New York, 1968.
[8] E.E. Moise, An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua, Trans. Amer. Math. Soc. 63 (1948) 581–594.
[9] S.B. Nadler Jr., Hyperspaces of Sets, Marcel Dekker, New York, 1978.
[10] D.R. Read, Confluent and related mappings, Colloq. Math. 29 (1974) 233–239.
[11] M. Smith, A hereditarily indecomposable Hausdorff continuum with exactly two composants, Topology Proc. 9 (1984) 123–143.
[12] R.W. Wardle, On a property of J.L. Kelley, Houston J. Math. 3 (1977) 291–299.