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GENERAL TOPOLOGY

Hyperspaces and the Property of Kelley

by

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Summary. An example of a continuum X is shown such that X and $C(X)$ has, while 2^X has not, the property of Kelley. This answers some two questions of Nadler.

Introduction. In [1], (16.35), p. 558, Sam B. Nadler, Jr. has shown an example of a continuum X having the property of Kelley and such that $X \times X$ fails to have the property. After this he asks some questions in connections with the property of Kelley for X and for the hyperspaces $C(X)$ and 2^X ([1], (16.37), p. 558). Two of them are: If X has the property of Kelley, then does 2^X have it, too? If $C(X)$ has the property of Kelley, then does 2^X have it, too? The example mentioned above is such that X has the property of Kelley, also $C(X)$ has it, but 2^X does not have the property. The aim of this paper is to show these two facts concerning the hyperspaces. The proof of the latter one is very similar to the proof that $X \times X$ fails to have the property of Kelley, given in [2]. Thus only two questions remain open in this area: 1° Is it true that if X has the property of Kelley, then $C(X)$ has also this property? 2° Is it true that if 2^X has the property of Kelley, then $C(X)$ has it, too? Note that an affirmative answer to 1° implies the same for 2° (by [1], (16.37), a remark on p. 559).

Basic notions. The letter X will denote a continuum (i.e. a compact connected metric space) with a metric ϱ. The hyperspace 2^X consists of all non-empty closed subsets $A \subset X$ with the Hausdorff metric $\operatorname{dist}(A, B) = \inf(\{\varepsilon > 0\colon A \subset N(B, \varepsilon) \text{ and } B \subset N(A, \varepsilon)\})$, where $N(A, \varepsilon)$ is the union of the ε-balls about all points of A. The hyperspace $C(X) \subset 2^X$ consists of all non-empty subcontinua of X and inherits the metric dist. We shall also be considering the hyperspace $C(2^X)$ with the Hausdorff metric denoted Dist. The symbol $\operatorname{diam}(A)$, for $A \in 2^X$, denotes the diameter of the set A, $\operatorname{bd}(A)$ means the boundary of A, and $\operatorname{cl}(A)$ means the closure of A.

We shall say that a continuum X has the property of Kelley if it satisfies the condition: given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $a, b \in X$ with $\varrho(a,b) < \delta$ and $a \in A \in C(X)$, then there is a continuum B with $b \in B \in C(X)$ and $\operatorname{dist}(A,B) < \varepsilon$.

The letter R will denote the real line, and H the half-line $[1, \infty) \subset R$.

Example. Now we prove that

(*) There exists a continuum X having the property of Kelley and such that the hyperspace $C(X)$ does have, while 2^X does not have, this property.

Define functions g and f mapping H into R^2 by

$$g(t) = \left(1 - \frac{1}{t}\right) e^{-it}, f(t) = \left(1 + \frac{1}{t}\right) e^{it}$$

and let $L = g(H)$ and $M = f(H)$. Denote by S the usual unit circle in R^2. The space $X = L \cup S \cup M$ is a continuum in R^2 (see [2], 4.7, p.297). It can be observed that X has the property of Kelley.

First we shall show that also $C(X)$ has this property. Denote by Cone(Y) the cone over a continuum Y. It is known (see [1], (8.23), p. 322) that there are homeomorphisms

$$h_L : \text{Cone}(L \cup S) \to C(L \cup S) \text{ and } h_M : \text{Cone}(M \cup S) \to C(M \cup S).$$

Thus it follows from [1], (8.26), p. 324, that there is an embedding $e\colon \text{Cone}(X) \to C(X)$. Note that $e(\text{Cone}(X)) = C(L \cup S) \cup C(M \cup S)$. and, by (1) of [1], (8.26), p. 324, that $e(v) = S$, where v is the vertex of Cone(X).

Consider now a family $\mathscr{A}$ of all these subcontinua of X that contain the limit circle S. Shrinking S to a point we can see that the family $\mathscr{A}$ is homeomorphic to the family of all subcontinua of a closed interval which contain its center (that is the image of S under the shrinking), and this family is homeomorphic to a disk D ([1], (8.23), p. 322). Common points of $\mathscr{A}$ and $C(L \cup S) \cup C(M \cup S)$ are exactly these continua K that $S \subset K$ and either $K \cap L = \emptyset$ or $K \cap M = \emptyset$. The continua K are points at which $\mathscr{A}$ and $C(L \cup S) \cup C(M \cup S)$ are locally connected (see [1], (1.132) and (1.133), p. 153). So we can see that the hyperspace $C(X)$ is homeomorphic to the union of Cone(X) and D with some points of local connectedness identified, whence it has the property of Kelley.

Now, assume on the contrary that 2^X also has this property. Then, given any $\varepsilon, 0 < \varepsilon < 1/2$, there exists a number $\delta, 0 < \delta < \varepsilon$, satisfying the definition of the property. Let $\mathscr{F}$ denote the set of all singletons in 2^X. Since $\mathscr{F}$ is homeomorphic to X, it is a subcontinuum of 2^X. Let $p \in L$ and $q \in M$ be two points such that $\varrho(p,q) < \delta$. Then $\operatorname{dist}(\{p,q\}, \{p\}) = \varrho(p,q) < \delta$ and by assumption there exists a continuum $\mathscr{K} \in C(2^X)$ containing $\{p,q\}$ with

$$\text{Dist}(\mathscr{K}, \mathscr{F}) < \varepsilon. \tag{1}$$

Note some properties of the continuum $\mathscr{K}$.

(2) If $A \in \mathscr{K}$, then $\operatorname{diam}(A) < 2\varepsilon$.

Really, let $x, y \in A$. By (1) there exists a point $z \in X$ such that $\operatorname{dist}(\{x,y\},\{z\}) < \varepsilon$, hence $\varrho(x,z) < \varepsilon$ and $\varrho(y,z) < \varepsilon$, and therefore $\varrho(x,y) \leqslant \varrho(x,z) + \varrho(y,z) < 2\varepsilon$.

Put $\mathscr{T} = \{A \in 2^X : A \cap L \neq \emptyset \neq A \cap M\}$ and note $\{p,q\} \in \mathscr{K} \cap \mathscr{T}$.

(3) $\mathscr{K}$ meets the complement of $\mathscr{T}$.

In fact, by (1), there exists an element A of $\mathscr{K}$ such that

$$\operatorname{dist}(A, \{g(1)\}) < \varepsilon, \text{ i.e., } A \subset N(\{g(1)\}, \varepsilon) \subset N(\{g(1)\}, 1/2) \subset L.$$

Define a mapping $l: \mathscr{T} \to H$ putting $l(A) = \min g^{-1}(A)$ and similarly define a mapping $m: \mathscr{T} \to H$ by $m(A) = \min f^{-1}(A)$. Let $\mathscr{L}$ be the component of $\mathscr{K} \cap \mathscr{T}$ containing the point $\{p,q\} \in \mathscr{K} \cap \mathscr{T}$. Thus $\mathscr{L}$ has a limit point in $\operatorname{bd}(\mathscr{K} \cap \mathscr{T})$, i.e., there exists a sequence of points A_n of $\mathscr{L}$ such that either $l(A_n) \to \infty$ or $m(A_n) \to \infty$ as $n \to \infty$.

Consider a function $\arg(A) = l(A) + m(A) - l(\{p,q\}) - m(\{p,q\})$. It is a continuous function from $\mathscr{L}$ into R, and $\arg(\{p,q\}) = 0$. Note that $\arg(A_n) \to \infty$ as $n \to \infty$, so the image of $\mathscr{L}$ under the mapping arg is an unbounded from the right and connected set in R containing the point 0. Thus there exists a set A_0 in $\mathscr{L}$ such that

$$\arg(A_0) = \pi. \tag{4}$$

Denote $a = l(A_0)$, $b = m(A_0)$, $c = l(\{p,q\})$ and $d = m(\{p,q\})$. Put $x = g(a)$ and $y = f(b)$, and note $x, y \in A_0$. Observe that $g(c) = p$ and $f(d) = q$. By (4) and by the definition of arg we have

$$a + b = c + d + \pi. \tag{5}$$

Now

$$\varrho(p,q) = [(1-1/c)^2 + (1+1/d)^2 - 2(1-1/c)(1+1/d)\cos(c+d)]^{1/2}.$$

If $\cos(c+d) < 0$, then all terms in the square brackets are positive, and therefore $\varrho(p,q) > 1$, a contradiction to the assumption $\varrho(p,q) < \delta < \varepsilon < 1/2$. Thus $\cos(c+d) \geqslant 0$. Consider now $\varrho(x,y)$.

$$\varrho(x,y) = [(1-1/a)^2 + (1+1/b)^2 - 2(1-1/a)(1+1/b)\cos(a+b)]^{1/2},$$

but $\cos(a+b) = -\cos(c+d) \leqslant 0$ by (5), and therefore $\varrho(x,y) > 1$ is contrary to (2) and to the choice of $\varepsilon < 1/2$. So (*) is shown.

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[2] R. W. Wardle, *On a property of J. L. Kelley*, Houston J. Math., 3 (1977), 291–299.

В. Я. Харатоник, Гиперпространства и свойство Келли

В настоящей работе приводится пример такого континуума X, что X и $C(X)$ имеют, а 2^X не имеет свойства Келли. Этот пример является ответом на два вопроса Надлера.