

**KERNELS OF HEREDITARILY UNICOHERENT
CONTINUA AND ABSOLUTE RETRACTS**

JANUSZ J. CHARATONIK, WŁODZIMIERZ J. CHARATONIK,
AND JANUSZ R. PRAJS

ABSTRACT. For a hereditarily unicoherent continuum X , its kernel means the common part of all subcontinua of X that intersect all arc components of X . This concept naturally appears when absolute retracts for the class of hereditarily unicoherent continua are studied. Let Y be such an absolute retract. Among other results, we prove that (a) Y is indecomposable if and only if it is identical with its kernel; (b) the dimension and the shape of Y are the same as ones of the kernel of Y ; (c) either Y is tree-like or the kernel of Y is indecomposable.

1. INTRODUCTION

The class of hereditarily unicoherent continua and their various subclasses, e.g., tree-like continua, λ -dendroids, dendroids, and dendrites, appear in a natural way in various regions of mathematical interest: the fixed point property, homogeneous spaces, continuous and upper semi-continuous decompositions, (hereditarily) indecomposable continua, and many other areas of topology, and also out of topology. These classes are hereditary and they

2000 *Mathematics Subject Classification.* Primary 54F15, 54F50. Secondary 54C55.

Key words and phrases. absolute retract, arc approximation property, arc component, arc property of Kelley, continuum, decomposable, dendroid, hereditarily unicoherent, kernel, property of Kelley, retraction.

The third named author was supported in part by Texas ARP grant No. 003644-012.

have many invariant properties with respect to numerous classes of mappings. They proved to be important and are among the most extensively studied classes of continua. By these reasons, investigation of absolute retracts for the mentioned classes of continua is both interesting and important. This paper is inspired by the study of absolute retracts for hereditarily unicoherent continua (the class of all such absolute retracts we denote by $\text{AR}(\mathcal{HU})$), and it is a continuation of the research initiated by Maćkowiak in [14]. Though continua in $\text{AR}(\mathcal{HU})$ need not be locally connected, see [14, corollaries 4 and 5, pp. 181 and 183], they all have the property of Kelley and the arc approximation property (see [4, Corollary 3.7]), and thus they have a very rich family of arcwise connected subcontinua. This observation leads to the conclusion that the *kernel* of a continuum $X \in \text{AR}(\mathcal{HU})$, defined as the intersection of all subcontinua of X that intersect all arc components of X , is a crucial concept to understand the structure of X . Though many results of this paper are formulated for classes of spaces larger than $\text{AR}(\mathcal{HU})$, the main idea of the paper is to study continua in $\text{AR}(\mathcal{HU})$ from the point of view of their kernels.

The paper consists of four sections. After the Introduction, some auxiliary concepts and results are collected in the second section. In section 3, properties of kernels of hereditarily unicoherent continua are studied. In particular, a characterization is obtained in terms of kernels of decomposable continua that have all arc components dense (Theorem 3.13). In the last section, we investigate kernels of hereditarily unicoherent continua with the arc property of Kelley. The obtained results are applied to study the classes of absolute retracts for hereditarily unicoherent continua, tree-like continua, and λ -dendroids in the case when the continuum X is not arcwise connected. It is shown (Theorem 4.6) that if a hereditarily unicoherent continuum has the generalized ε -push property and contains a dense arc component, then either it is tree-like or its kernel is indecomposable.

This paper is mainly devoted to study general properties of kernels of continua in $\text{AR}(\mathcal{HU})$. In a forthcoming paper the authors will show that each tree-like continuum is the kernel of some element of $\text{AR}(\mathcal{HU})$. In particular, according to this announcement, there is a large family of continua studied in the present paper.

2. PRELIMINARIES

By a *space* we mean a topological space, and a *mapping* means a continuous function. The reader is referred to [2] and [9] for needed information on the concepts of a retraction and a retract.

Let \mathcal{C} be a class of *compacta*, i.e., of (nonempty) compact metric spaces. Following [9, p. 80], we say that a space $Y \in \mathcal{C}$ is an *absolute retract for the class \mathcal{C}* (abbreviated $\text{AR}(\mathcal{C})$) if for any space $Z \in \mathcal{C}$ such that Y is a subspace of Z , Y is a retract of Z . The concept of an AR space originally had been studied by Borsuk [2].

Let X be a metric space with a metric d . For a mapping $f : A \rightarrow B$, where A and B are subspaces of X , we define $d(f) = \sup\{d(x, f(x)) : x \in A\}$. Further, we denote by $B(p, \varepsilon)$ the (open) ball in X centered at a point $p \in X$ and having the radius ε . For a subset $A \subset X$, we put $N(A, \varepsilon) = \bigcup\{B(a, \varepsilon) : a \in A\}$. The symbol \mathbb{N} stands for the set of all positive integers, and \mathbb{R} denotes the space of real numbers.

By a *continuum* we mean a connected compactum. Given a continuum X , let $C(X)$ be the hyperspace of all nonempty subcontinua of X equipped with the Hausdorff metric (see e.g., [16, (0.1), p. 1 and (0.12), p. 10]). The reader is referred to [13] and [16] for concepts not defined here.

Let \mathcal{D}_0 denote the class of dendrites, \mathcal{D} — the class of dendroids, $\lambda\mathcal{D}$ — of λ -dendroids, \mathcal{TL} — of tree-like continua, and \mathcal{HU} — the class of hereditarily unicoherent ones. Then

$$(2.1) \quad \mathcal{D}_0 \subset \mathcal{D} \subset \lambda\mathcal{D} \subset \mathcal{TL} \subset \mathcal{HU}.$$

According to a classical result of Borsuk [2, (13.5), p. 138], each dendrite is an absolute retract for the class of all compacta. Consequently, any dendrite D is an absolute retract for each class \mathcal{C} of compacta (abbreviated $\text{AR}(\mathcal{C})$) such that $D \in \mathcal{C}$. More generally, if $\mathcal{C}_1 \subset \mathcal{C}_2$ for some classes \mathcal{C}_1 and \mathcal{C}_2 of spaces, then

$$\mathcal{C}_1 \cap \text{AR}(\mathcal{C}_2) \subset \text{AR}(\mathcal{C}_1).$$

In particular, we have

$$\text{AR}(\mathcal{D}_0) = \mathcal{D}_0 \subset \text{AR}(\mathcal{D}) \cap \text{AR}(\lambda\mathcal{D}) \cap \text{AR}(\mathcal{TL}) \cap \text{AR}(\mathcal{HU}).$$

Note that the class of absolute retracts of all unicoherent continua coincides with the class of retracts of the Hilbert cube; thus, it also coincides with the class of absolute retracts of all compacta.

This class is relatively well studied, and we do not investigate it here.

In the rest of this section, we collect concepts and results used in section 4, mostly introduced and studied in our very recent papers, and therefore perhaps not known to the reader. The aim of this section is to support the reader in understanding our arguments applied in proofs of results in the next two sections.

We start with recalling the following concept formulated in [4, Definition 2.20]. A continuum X is said to have the *generalized ε -push property* provided that

$$(2.2) \text{ for each } \varepsilon > 0 \text{ there is a } \delta > 0 \text{ such that for every two points } x, y \in X \text{ with } d(x, y) < \delta \text{ there exists a mapping } f : X \rightarrow X \text{ satisfying } f(x) = y \text{ and } d(f) < \varepsilon.$$

The next result is shown as [4, Corollary 2.22].

Proposition 2.3. *Let $\mathcal{K} \in \{\mathcal{D}, \lambda\mathcal{D}, \mathcal{TL}, \mathcal{HU}\}$. Then each member of $\text{AR}(\mathcal{K})$ has the generalized ε -push property.*

A continuum X is said to have the *property of Kelley* provided that for each point $p \in X$, for each subcontinuum K of X containing p , and for each sequence of points p_n converging to p there exists a sequence of subcontinua K_n of X containing p_n and converging (with respect to the Hausdorff metric) to the continuum K (see e.g., [16, Definition 16.10, p. 538]).

The next result has been established in [4, Proposition 3.1].

Proposition 2.4. *Each continuum having the generalized ε -push property has the property of Kelley.*

A continuum X is said to have the *arc approximation property* provided that for each point $x \in X$, for each subcontinuum K of X containing x there exists a sequence of arcwise connected subcontinua K_n of X containing x and converging to the continuum K (see [6, Section 3, p. 113]). The next proposition is known, see [6, Proposition 3.10, p. 116].

Proposition 2.5. *If a continuum has the arc approximation property, then each arc component of the continuum is dense.*

A continuum X is said to have the *arc property of Kelley* (see [4, Definition 3.3]) provided that for each point $p \in X$, for each subcontinuum K of X containing p , and for each sequence of points p_n converging to p there exists a sequence of arcwise connected subcontinua K_n of X containing p_n and converging to the continuum K . The next results are shown in [4, Proposition 3.4 and Proposition 3.7].

Proposition 2.6. *A continuum has the arc property of Kelley if and only if it has the arc approximation property and the property of Kelley.*

Proposition 2.7. *Let \mathcal{K} be any class of continua listed in (2.1). Then any member of $\text{AR}(\mathcal{K})$ has the arc property of Kelley.*

The reader is referred to [4] for some results more general than quoted above.

3. KERNELS

Investigating absolute retracts for various classes of hereditarily unicoherent continua, the concept of a kernel defined below appears in a natural manner. We define it in a more general setting, namely for all hereditarily unicoherent continua. Next we prove some general properties of kernels, and then we investigate kernels of those continua in the class \mathcal{HU} which have more specific properties, such as dense arc component(s), the arc approximation property, the (arc) property of Kelley, and the generalized ε -push property (see (2.2)).

Definition 3.1. Let X be a hereditarily unicoherent continuum, and let $\mathcal{F}(X)$ be the family of all subcontinua of X intersecting all arc components of X . The intersection of all members of the family $\mathcal{F}(X)$ is named the *kernel* of X , and is denoted by $\text{Ker}(X)$. Since a continuum X is hereditarily unicoherent if and only if the intersection of all members of any family of subcontinua of X is a continuum, the kernel $\text{Ker}(X)$ is a subcontinuum of X .

Theorem 3.2. *For every two hereditarily unicoherent, not arcwise connected continua X and Y if $X \subset Y$, then $\text{Ker}(X) \subset \text{Ker}(Y)$.*

Proof: Note that for any $A \in \mathcal{F}(Y)$, the intersection $A \cap X$ belongs to $\mathcal{F}(X)$ by the hereditary unicoherence of Y . Thus

$$\text{Ker}(X) = \bigcap \mathcal{F}(X) \subset \bigcap \{A \cap X : A \in \mathcal{F}(X)\} \subset \bigcap \mathcal{F}(Y) = \text{Ker}(Y).$$

□

Proposition 3.3. *If subcontinua A and B of a hereditarily unicoherent not arcwise connected continuum X intersect all arc components of X , then $A \cap B \neq \emptyset$, and the continuum $A \cap B$ intersects all arc components of X .*

Proof: Let L_1 and L_2 be arcs in two different arc components of X , both joining A and B . If $A \cap B = \emptyset$, the union $L_1 \cup L_2 \cup A \cup B$ is not unicoherent, a contradiction. Thus, $A \cap B \neq \emptyset$.

If $L_1 \cap A \cap B = \emptyset$, then the union $L_1 \cup A \cup B$ is not unicoherent, a contradiction. Therefore, $A \cap B$ intersects each arc component of X . □

Corollary 3.4. *Let X be a nondegenerate hereditarily unicoherent continuum. Then X is arcwise connected (i.e., X is a dendroid) if and only if $\text{Ker}(X) = \emptyset$.*

If a closed subset C of a continuum X is given, then X/C is the quotient space obtained by shrinking C to a point. Thus, if C is a continuum, the quotient mapping $q : X \rightarrow X/C$ is monotone. See [18, Chapter 7, p. 122] for the details.

Theorem 3.5. *If X is a hereditarily unicoherent, not arcwise connected continuum, then $\text{Ker}(X)$ is the smallest continuum Y such that X/Y is arcwise connected (i.e., X/Y is a dendroid).*

Proof: First we show that

$$(3.5.1) \quad X/\text{Ker}(X) \text{ is a dendroid.}$$

By Proposition 3.3, there is a decreasing sequence of continua Y_1, Y_2, \dots , each of which intersects all components of X and such that $\text{Ker}(X) = \bigcap \{Y_n : n \in \mathbb{N}\}$. Then, for each $n \in \mathbb{N}$, there is a natural monotone mapping $m_n : X/Y_{n+1} \rightarrow X/Y_n$. Therefore, the quotient space $X/\text{Ker}(X) = \varprojlim \{X/Y_n, m_n\}$ is the inverse limit of dendroids X/Y_n with monotone bonding mappings m_n , so it is a

dendroid, see [15, Theorem 4, p. 229] and [1, Lemma 1, p. 192]. Thus, (3.5.1) is shown.

Let $\mathcal{G}(X) = \{C \in C(X) : X/C \text{ is arcwise connected}\}$ and

$$A(C, \varepsilon) = C \cup \bigcup \{A \subset N(C, \varepsilon) : A \text{ is an arc and } A \cap C \neq \emptyset\}.$$

Observe that if $C \in \mathcal{G}(X)$, then $\text{cl}A(C, \varepsilon) \in \mathcal{F}(X)$ for each $\varepsilon > 0$. Therefore, if $q \notin C$ for some $C \in \mathcal{G}(X)$, then $q \notin \bigcap \mathcal{F}(X) = \text{Ker}(X)$, and thus $\text{Ker}(X) \subset \bigcap \mathcal{G}(X)$. \square

Corollary 3.6. *For each hereditarily unicoherent, not arcwise connected continuum X , we have $\dim X = \dim \text{Ker}(X)$.*

Proof: Using [11, §25, II, Theorem 1, p. 274 and §27, I, corollaries 2a and 2c, p. 289], we infer that $\dim X = \max\{\dim \text{Ker}(X), \dim (X/\text{Ker}(X))\}$. Since $X/\text{Ker}(X)$ is a dendroid according to Theorem 3.5, its dimension is one, [3, (48), p. 239], so the conclusion follows. \square

In the next part of this section we use the concept of the shape of a compact metric space as defined in e.g., [8].

Theorem 3.7. *If Y is a subcontinuum of a continuum X and X/Y is a tree-like continuum, then $\text{Sh}(X) = \text{Sh}(Y)$.*

Proof: Recall that a *finite tree cover* of a space means a finite acyclic cover such that no three of its elements intersect.

Embed X in the Hilbert cube \mathbb{Q} . We start by proving two claims.

CLAIM 1. *For each $\varepsilon > 0$, there exists a finite tree cover $\{U_1, U_2, \dots, U_k\}$ of X such that :*

- 1) *all U_i for $i \in \{1, 2, \dots, k\}$ are compact subsets of the Hilbert cube \mathbb{Q} ;*
- 2) *for $i \in \{2, \dots, k\}$, the sets U_i are homeomorphic to \mathbb{Q} ;*
- 3) *$Y \subset \text{int } U_1 \subset U_1 \subset N(Y, \varepsilon)$;*
- 4) *$\text{diam } U_i < \varepsilon$ for $i \in \{2, \dots, k\}$;*
- 5) *$X \subset \text{int } (\bigcup \{U_i : i \in \{1, 2, \dots, k\}\})$.*

To prove Claim 1, consider first a finite tree cover \mathcal{U}_1 of X/Y of a sufficiently small mesh. Second, take the cover \mathcal{U}_2 of X consisting of the preimages under the quotient mapping defined on the Hilbert cube \mathbb{Q} and shrinking Y to a point. Finally, note that there is a refinement of \mathcal{U}_2 satisfying all the needed conditions 1)-5).

In the next claim we use the notation in Claim 1.

CLAIM 2. *There is a homotopy*

$$H : (U_1 \cup U_2 \cup \dots \cup U_k) \times [0, 1] \rightarrow U_1 \cup U_2 \cup \dots \cup U_k$$

such that

$$\begin{aligned} H(x, 0) &= x, \quad H(x, 1) \in U_1, \quad \text{and} \\ H(x, t) &= x \quad \text{for } x \in U_1 \quad \text{and each } t \in [0, 1]. \end{aligned}$$

Indeed, Claim 2 follows from the fact that the sets U_2, \dots, U_k are homeomorphic to \mathbb{Q} and that $\{U_1, U_2, \dots, U_k\}$ is a tree cover.

We will construct two inverse sequences $\{X_n, f_n\}$ and $\{Y_n, g_n\}$, and embeddings $e_n : Y_n \rightarrow X_n$ such that all mappings in the diagram

$$(*) \quad \begin{array}{ccccccc} Y_1 & \xleftarrow{g_1} & Y_2 & \xleftarrow{g_2} & Y_3 & \xleftarrow{g_3} & \dots & Y \\ e_1 \downarrow & & e_2 \downarrow & & e_3 \downarrow & & & e \downarrow \\ X_1 & \xleftarrow{f_1} & X_2 & \xleftarrow{f_2} & X_3 & \xleftarrow{f_3} & \dots & X \end{array}$$

are embeddings, and thus

$$X = \bigcap \{X_n : n \in \mathbb{N}\} = \varprojlim \{X_n, f_n\},$$

$$Y = \bigcap \{Y_n : n \in \mathbb{N}\} = \varprojlim \{Y_n, g_n\},$$

and $e = \varprojlim e_n : Y \rightarrow X$ is an embedding, too.

To do so, take $\varepsilon_1 = 1$, put $Y_1 = U_1$ and $X_1 = U_1 \cup U_2 \cup \dots \cup U_k$ as in Claim 1, and let $e_1 : Y_1 \rightarrow X_1$ be the embedding. Let ε_2 be a positive number such that $N(X, \varepsilon_2) \subset X_1$ and $N(Y, \varepsilon_2) \subset Y_1$. Use Claim 1 again to construct X_2 and Y_2 , as previously. Continuing in this way, we get the needed sequences in (*).

For each $n \in \mathbb{N}$ let $H_n : X_n \times [0, 1] \rightarrow X_n$ be the homotopy as in Claim 2. Define a mapping $h_n : X_n \rightarrow Y_n$ by $h_n(x) = H_n(x, 1)$ for each $x \in X_n$. Then the diagram

$$\begin{array}{ccccccc} Y_1 & \xleftarrow{g_1} & Y_2 & \xleftarrow{g_2} & Y_3 & \xleftarrow{g_3} & \dots & Y \\ e_1 \updownarrow h_1 & & e_2 \updownarrow h_2 & & e_3 \updownarrow h_3 & & & e \downarrow \\ X_1 & \xleftarrow{f_1} & X_2 & \xleftarrow{f_2} & X_3 & \xleftarrow{f_3} & \dots & X \end{array}$$

commutes up to homotopy, i.e., all the respective compositions are homotopic. In fact, $h_n \circ e_n$ is the identity on Y_n , and $e_n \circ h_n = h_n$ is homotopic to the identity on X_n . The mappings h_1, \dots, h_n, \dots induce a shape morphism $\underline{h} : X \rightarrow Y$ which is the inverse of the embedding $e : Y \rightarrow X$. Thus $\text{Sh}(X) = \text{Sh}(Y)$ as needed. \square

As a consequence of Theorem 3.7, we get the following corollaries.

Corollary 3.8. *Let X be a hereditarily unicoherent, not arcwise connected continuum. Then $\text{Sh}(\text{Ker}(X)) = \text{Sh}(X)$.*

Corollary 3.9. *Let a continuum X be in $\text{AR}(\mathcal{HU})$. Then X is tree-like if and only if $\text{Ker}(X)$ is tree-like.*

Theorem 3.10. *For each hereditarily unicoherent, not arcwise connected continuum X , the kernel $\text{Ker}(X)$ contains all nondegenerate indecomposable subcontinua of X .*

Proof: Let M be a nondegenerate indecomposable subcontinuum of X . Since arc components of M are contained in its composants, M is the only subcontinuum of M intersecting all these arc components. Thus, $M = \text{Ker}(M) \subset \text{Ker}(X)$ by Theorem 3.2. \square

A subcontinuum T of a continuum X is said to be *terminal* in X provided that for each subcontinuum K of X the condition $K \cap T \neq \emptyset$ implies $K \subset T$ or $T \subset K$. Note that, according to the definition, the whole continuum X is a terminal subcontinuum of itself, and that each singleton is terminal.

Theorem 3.11. *For each hereditarily unicoherent, not arcwise connected continuum X , the kernel $\text{Ker}(X)$ contains all proper nondegenerate terminal subcontinua of any continuum $Y \subset X$.*

Proof: Let T be a proper nondegenerate terminal subcontinuum of Y . Then no arc contained in Y intersects both T and $Y \setminus T$, and thus each subcontinuum of Y intersecting all arc components of Y contains T . Then $T \subset \text{Ker}(Y)$, and since $\text{Ker}(Y) \subset \text{Ker}(X)$ by Theorem 3.2, the conclusion follows. \square

Proposition 3.12. *If X is a hereditarily unicoherent, not arcwise connected continuum having all its arc components dense, and if Y is a subcontinuum of X such that $X \setminus Y$ is not connected, then Y intersects all arc components of X , and consequently $\text{Ker}(X) \subset Y$.*

Proof: Let A and B be two disjoint open subsets of $X \setminus Y$ such that $X \setminus Y = A \cup B$, and let C be an arc component of X . Then $A \cap C \neq \emptyset \neq B \cap C$, whence $Y \cap C \neq \emptyset$ by connectedness of C . \square

Theorem 3.13. *Let X be a hereditarily unicoherent, not arcwise connected continuum having all arc components dense. Then X is decomposable if and only if $\text{Ker}(X)$ is a proper subcontinuum of X .*

Proof: If X is indecomposable, then the only subcontinuum of X intersecting all arc components of X is X itself, so $\text{Ker}(X) = X$.

Assume X is decomposable. Let $X = A \cup B$, where A and B are proper subcontinua of X . We will show that $A \cap B$ intersects all arc components of X . Let C be an arbitrary arc component of X . Since C is dense, $A \cap C \neq \emptyset \neq B \cap C$; therefore, by hereditary unicoherence of X , we get $(A \cap B) \cap C \neq \emptyset$, as needed. Thus, $\text{Ker}(X) \subset A \cap B \neq X$. \square

As a consequence of Theorem 3.13 and Proposition 2.5, we get the next result.

Theorem 3.14. *Let X be a hereditarily unicoherent, not arcwise connected continuum having the arc approximation property. Then X is decomposable if and only if $\text{Ker}(X)$ is a proper subcontinuum of X .*

4. KERNELS OF CONTINUA IN $\text{AR}(\mathcal{HU})$

In this section we will study properties of kernels of continua in $\text{AR}(\mathcal{HU})$. Our results will be formulated even in a more general way, namely for kernels of hereditarily unicoherent continua with the arc property of Kelley or with the generalized ε -push property. Since each absolute retract for any of the studied classes of continua in \mathcal{HU} has all properties mentioned in the beginning of Section 3, all results of this section can be applied to such absolute retracts.

Recall that a mapping $f : X \rightarrow Y$ between continua X and Y is said to be *confluent* provided that for each continuum Q in Y , each component of $f^{-1}(Q)$ is mapped onto Q under f . Obviously, any monotone mapping is confluent.

Theorem 4.1. *Let X be a hereditarily unicoherent, not arcwise connected continuum having the arc property of Kelley. Then*

$X/\text{Ker}(X)$ is a dendroid having the property of Kelley, so it is smooth, and $\text{Ker}(X)$ is the only point of local connectedness and the only initial point of $X/\text{Ker}(X)$.

Proof: By Theorem 3.5, the continuum $D = X/\text{Ker}(X)$ is a dendroid. Since confluent mappings preserve the property of Kelley, [17, Theorem 4.3, p. 296], D , as a monotone image of X under the quotient mapping $q : X \rightarrow X/\text{Ker}(X) = D$, has the property of Kelley, so it is smooth according to [7, Corollary 5, p. 730]. Recall that a smooth dendroid is locally connected at each of its initial points, see [5, Theorem 2, p. 299], and therefore it is enough to prove that $\text{Ker}(X)$ is the only point of local connectedness of D .

Assume that $p \in D$ is a point of local connectedness of D and suppose that $p \neq q(\text{Ker}(X))$. Since q is one-to-one on $X \setminus \text{Ker}(X)$, the point $q^{-1}(p)$, at which X is locally connected, has arbitrarily small arcwise connected neighborhoods. This means that arc components of X , not containing $q^{-1}(p)$, are not dense in X , contrary to Proposition 2.5. □

Theorem 4.2. *Let X be a hereditarily unicoherent, not arcwise connected continuum having the arc property of Kelley. Then each subcontinuum Y of X such that $Y \cap \text{Ker}(X) = \emptyset$ is a nowhere dense smooth dendroid.*

Proof: Let again $q : X \rightarrow X/\text{Ker}(X) = D$ be the quotient mapping. By Theorem 4.1 the continuum D is a smooth dendroid. Thus, $q(Y)$ as a subset of D is a smooth dendroid [5, Corollary 6, p. 299]. Since $q|_Y$ is a homeomorphism, Y is a smooth dendroid, too. To show that it is nowhere dense observe that, by Proposition 2.5 and by the arc property of Kelley, there are continua in other arc components of X which are close to Y . □

Theorem 3.5 gives a characterization of a kernel of a continuum. Below we present one more characterization assuming that the continuum has the arc property of Kelley.

Theorem 4.3. *Let X be a hereditarily unicoherent, not arcwise connected continuum having the arc property of Kelley. Then $\text{Ker}(X)$ is the intersection of all subcontinua Y of X such that $X \setminus Y$ is not connected. Moreover, there is a decreasing sequence of subcontinua Y_n of X such that $X \setminus Y_n$ is not connected for each $n \in \mathbb{N}$ and that $\text{Ker}(X) = \bigcap \{Y_n : n \in \mathbb{N}\}$.*

Proof: Denote by P the intersection of all subcontinua Y of X such that $X \setminus Y$ is not connected. Then P is a continuum by hereditary unicoherence of X . Further, $\text{Ker}(X) \subset P$ by Proposition 3.12. We will show the other inclusion.

Let $q : X \rightarrow X/\text{Ker}(X) = D$ be the quotient mapping. By Theorem 4.1 the continuum D is a smooth dendroid, and the point $v = q(\text{Ker}(X))$ is the only initial point of D . Consider a radially convex metric on D , (see [5, Corollary 11, p. 311]), and denote by L_n the closure of the $\frac{1}{n}$ -ball about v in this metric. Then L_n is a closed connected neighborhood of v with the 0-dimensional boundary. Consequently, $q^{-1}(L_n)$ is a subcontinuum of X such that $X \setminus q^{-1}(L_n)$ is homeomorphic to $D \setminus L_n$. We will show that $D \setminus L_n$ is not connected for almost all $n \in \mathbb{N}$. To this aim, consider two cases.

Case 1. $\text{bd}L_n$ is a one-point set for infinitely many indices $n \in \mathbb{N}$. Then $\text{bd}q^{-1}(L_n)$ is a one-point set (in X), contrary to density of arc components of X (see Proposition 2.5).

Case 2. $\text{bd}L_n$ has more than one point for almost all $n \in \mathbb{N}$. Since $\text{bd}L_n$ is 0-dimensional, we can write $\text{bd}L_n = B_n^1 \cup B_n^2$, where B_n^1 and B_n^2 are disjoint closed subsets of D . For $i \in \{1, 2\}$, denote by U^i the union of all components of $D \setminus L_n$ whose closure intersects B_n^i . Then $\text{cl } U^i = U^i \cup B_n^i$, so $D \setminus L_n$ is the union of two separated subsets U^1 and U^2 . Therefore, $D \setminus L_n$ is not connected, as needed.

Putting $Y_n = q^{-1}(L_n)$, we end the proof. \square

In the previous theorem we showed that the kernel is the intersection of continua with disconnected complements. The complement of the kernel itself may or may not be connected. The next theorem shows that if the complement of the kernel is not connected, then the kernel is indecomposable.

Theorem 4.4. *Let X be a hereditarily unicoherent, not arcwise connected continuum having the arc property of Kelley. If $X \setminus \text{Ker}(X)$ is not connected, then $\text{Ker}(X)$ is indecomposable.*

Proof: First recall that by Proposition 3.12, the continuum $\text{Ker}(X)$ intersects all arc components of X . Let $\text{Ker}(X) = P \cup Q$, where P and Q are proper subcontinua of $\text{Ker}(X)$. By Proposition 3.3, there is an arc component C of X disjoint with P or Q . Assume $C \cap Q = \emptyset$. Let U and V be two disjoint open subsets of X such that

$X \setminus \text{Ker}(X) = U \cup V$, and denote by $A = ab$ an arc in C such that $A \cap U \neq \emptyset \neq A \cap V$. By the arc property of Kelley of X there is a small open neighborhood W of the point a such that for each $x \in W$ there is an arc A_x such that $x \in A_x$ and $A_x \cap V \neq \emptyset = A_x \cap Q$. Then $A_x \cap P \neq \emptyset$, and since each arc component of X intersects U , by Proposition 2.5 it follows that each arc component of X intersects P . Therefore, $\text{Ker}(X) \subset P$, a contradiction. \square

The next example shows that the assumptions of Theorem 4.4 are satisfied by some continua.

Example 4.5. *There exists a hereditarily unicoherent, not arcwise connected continuum having the arc property of Kelley and such that $X \setminus \text{Ker}(X)$ is not connected; consequently, $\text{Ker}(X)$ is indecomposable.*

Proof: Let B be the buckethandle continuum (called also the simplest Brouwer-Janiszewski-Knaster indecomposable continuum) situated in the plane in its standard position as defined in [12, §48, V, Example 1, Figure 4, pp. 204–205], and let C be the standard Cantor set obtained as the intersection of B and the closed unit interval $[0, 1]$ of the x -axis. For each point $p \in C$, let I_p be an arc such that $I_p \cap B = \{p\}$ and that the union $\bigcup\{I_p : p \in C\}$ is homeomorphic to $C \times [0, 1]$. Then $X = B \cup \bigcup\{I_p : p \in C\}$ is the needed continuum. Observe that $\text{Ker}(X) = B$ and that $X \setminus \text{Ker}(X)$ is homeomorphic to $C \times (0, 1]$, so it is not connected. The required properties of X are straightforward. \square

We do not know if the constructed continuum X belongs to $\text{AR}(\mathcal{HU})$. Using some results that will be presented in a future paper we can, however, obtain a similar example that is a member of $\text{AR}(\mathcal{HU})$.

The principal application of the above presented results concerning kernels is an investigation of the classes $\text{AR}(\mathcal{HU})$, $\text{AR}(\mathcal{TL})$ and $\text{AR}(\lambda\mathcal{D})$ in the case when the continuum X is not arcwise connected. Indeed, all these continua have nondegenerate kernels, and they have the arc property of Kelley. According to the results presented by the authors in a forthcoming paper, the family $\text{AR}(\mathcal{TL})$ is large, and any tree-like continuum can be the kernel of a member of $\text{AR}(\mathcal{TL})$. But we do not have any examples belonging to

$\text{AR}(\mathcal{HU}) \setminus \text{AR}(\mathcal{TL})$. The sequential results are motivated by studying just this class.

Theorem 4.6. *If a continuum X in \mathcal{HU} has the generalized ε -push property and contains a dense arc component, then either X is tree-like or $\text{Ker}(X)$ is indecomposable.*

Proof: Let us observe that if X has a dense arc component, then condition (2.2) implies that all arc components of X are dense. Thus, in the light of Theorem 3.13, it is enough to show the conclusion if $\text{Ker}(X)$ is a proper subcontinuum of X .

Assume that $\text{Ker}(X)$ is decomposable, and let $\text{Ker}(X) = P_1 \cup P_2$, where P_1 and P_2 are proper subcontinua of $\text{Ker}(X)$. For any $\varepsilon > 0$ and for $i \in \{1, 2\}$ define $P_i(\varepsilon)$ as the closure of the component of $N(P_i, \varepsilon)$ that contains P_i .

CLAIM 1. *There are arc components A_1 and A_2 of X and there is $\varepsilon_0 > 0$ such that $A_i \cap P_i(\varepsilon_0) = \emptyset$ for each $i \in \{1, 2\}$.*

Indeed, otherwise for each $\varepsilon > 0$ there is $i \in \{1, 2\}$ such that $P_i(\varepsilon)$ intersects all arc components of X . Consequently, we get $\text{Ker}(X) \subset \bigcap \{P_i(\varepsilon) : \varepsilon > 0\} = P_i$, a contradiction.

CLAIM 2. *Let $i, j \in \{1, 2\}$ with $i \neq j$. For each subcontinuum K of X the condition $K \cap A_i \neq \emptyset \neq K \cap P_i$ implies $K \cap (P_j \setminus P_i) \neq \emptyset$.*

Suppose that the implication does not hold, and consider the irreducible continuum C between a point $a \in A_i$ and $p \in P_i$. Put $C_0 = C \cap P_i$. Then the quotient mapping $q : X \rightarrow X/\text{Ker}(X)$ is monotone; thus, it is hereditarily monotone (see [13, (6.10), p. 53]), and therefore $q|C$ is monotone. Hence, the image $q(C)$ is irreducible between $q(a)$ and $q(p) = q(\text{Ker}(X))$ (see [12, §48, I, Theorem 3, p. 192]), so it is an arc by Theorem 3.5. Thus, the set $S = C \setminus C_0$ is homeomorphic to the real half line, which lies in A_i , and some subcontinuum $C_i \subset C_0$ is the remainder of a compactification of S . Since $C_i \subset P_i$, the set $P_i(\varepsilon_0)$ intersects A_i , contrary to Claim 1.

CLAIM 3. *For $i \in \{1, 2\}$, any continuum contained in $X \setminus P_i$ is tree-like.*

Fix $i = 1$. Let K be a subcontinuum of $X \setminus P_1$. Choose $x \in K$. Since A_2 is dense in X , there is a sequence $\{a_n\}$ in A_2 converging to x . By Proposition 2.3, there is a sequence of mappings $f_n : X \rightarrow X$

such that $f_n(x) = a_n$ and $\lim d(f_n) = 0$. Then almost all sets $K_n = f_n(K)$ are disjoint with P_1 . They cannot intersect P_2 by Claim 2. Consequently, they are subsets of $X \setminus \text{Ker}(X)$, so they are dendroids, so tree-like. Thus, for each ε , the continuum K has an ε -mapping onto a tree-like continuum, whence it follows that K is tree-like.

The argument for $i = 2$ is the same.

CLAIM 4. *The continua P_1 and P_2 are tree-like.*

Let $p \in P_1$. Choose a sequence $\{a_n\}$ in A_1 with $\lim a_n = p$. By Proposition 2.3 there is a sequence of mappings $f_n : X \rightarrow X$ such that $f_n(p) = a_n$ and $\lim d(f_n) = 0$. Then $f_n(P_1) \cap P_1 = \emptyset$ for almost all indices n , because otherwise almost all unions $P_1 \cup f_n(P_1)$ were continua in $P_1(\varepsilon_0)$ joining P_1 and A_1 contrary to Claim 1. Thus by Claim 3 these continua $f_n(P_1)$ are tree-like. Thereby for each $\varepsilon > 0$, the continuum P_1 has ε -mappings onto tree-like continua, and therefore, it is tree-like. For P_2 , the proof is the same.

It follows from Claim 4 that $\text{Ker}(X) = P_1 \cup P_2$ is tree-like. Thus, $\text{Sh}(\text{Ker}(X))$ is trivial and $\dim \text{Ker}(X) = 1$, whence $\dim X = 1$ by Corollary 3.6. Further, by Proposition 2.4, the continuum X has the property of Kelley, whence it follows that $X/\text{Ker}(X)$ has the property as a monotone, thus confluent, image of X (see [17, Theorem 4.3, p. 296]). Since $X/\text{Ker}(X)$ is a dendroid according to Theorem 3.5, it is a dendroid having the property of Kelley; thus, it is smooth (see [7, Corollary 5, p. 730]). Hence, Theorem 3.7 implies that $\text{Sh}(X)$ is trivial. As a one-dimensional continuum having trivial shape, the continuum X is tree-like, see [10, Theorem 2.1, Part B, p. 237]. \square

Corollary 4.7. *If $X \in \text{AR}(\mathcal{HU})$, then either X is tree-like or $\text{Ker}(X)$ is indecomposable.*

Proof: The continuum X has the arc property of Kelley by Proposition 2.7, whence all assumptions of Theorem 4.6 are satisfied according to Propositions 2.6 and 2.5, so the conclusion follows. \square

Corollary 4.8. *If $X \in \text{AR}(\mathcal{HU}) \setminus \text{AR}(\mathcal{TL})$, then $\text{Ker}(X)$ is indecomposable.*

Theorem 4.9. *If a continuum X in \mathcal{HU} has the generalized ε -push property and contains a dense arc component, then at least one of the following two conditions is satisfied:*

- (4.9.1) *each subcontinuum K of X such that $\text{Ker}(X) \setminus K \neq \emptyset$ is tree-like;*
- (4.9.2) *for each arc component A of X we have $\text{cl}(A \cap \text{Ker}(X)) = \text{Ker}(X)$. In particular, each arc component of $\text{Ker}(X)$ is dense in $\text{Ker}(X)$.*

Proof: If X is tree-like, the conclusion obviously holds. So, assume that X is not tree-like. Then the kernel $\text{Ker}(X)$ is indecomposable according to Theorem 4.6. Suppose on the contrary that (4.9.2) does not hold, i.e., that there is an arc component A of X such that $L = \text{cl}(A \cap \text{Ker}(X)) \neq \text{Ker}(X)$. Since X is hereditarily unicoherent, it follows that $A \cap \text{Ker}(X)$ is connected. Thus, either L is a proper subcontinuum of X or $L = \emptyset$. In the latter case, we take a continuum K which is irreducible with respect to containing a point $a \in A$ and intersecting the kernel $\text{Ker}(X)$, and we put $K_0 = K \cap \text{Ker}(X)$.

CLAIM 1. *If a subcontinuum M of X intersects both A and $\text{Ker}(X)$, then:*

- (1a) *$L \neq \emptyset$ implies $L \cap M \neq \emptyset$;*
- (1b) *$L = \emptyset$ implies $K_0 \subset M$.*

Indeed, if $L \neq \emptyset$, then $A \cap \text{Ker}(X) \neq \emptyset$. Since X is hereditarily unicoherent, the continuum M contains a point in $A \cap \text{Ker}(X)$ which belongs to L , so (1a) holds.

If $L = \emptyset$, choose in M a continuum M_1 which is irreducible with respect to containing a point $a_1 \in A$ and intersecting $\text{Ker}(X)$, and define $M_0 = M_1 \cap \text{Ker}(X)$. Denote by I the arc in A joining the points a and a_1 . If $K_0 \cap M_0 = \emptyset$, then the continua $P = I \cup M_1$ and $Q = I \cup K$ have their intersection $P \cap Q$ nonempty and not connected, a contradiction with the hereditary unicoherence of X . Thus, $K_0 \cap M_0 \neq \emptyset$. Therefore, $K \cup M_1 \subset P \cap Q$ by the irreducibility of K and M_1 . Since $I \subset A \setminus \text{Ker}(X)$, hence $K_0 \cup M_0 \subset P \cap Q = K_0 \cap M_0$. Consequently, $K_0 = M_0$, and thereby $K_0 \subset M$, i.e., (1b) holds.

Thus, Claim 1 is shown.

Now we consider two cases.

Case A. Either $L \neq \emptyset$ or $K_0 \neq \text{Ker}(X)$.

Define $G = L$ if $L \neq \emptyset$, and $G = K_0$ if $L = \emptyset$. Under Case A we will show two claims.

CLAIM 2A. *For each subcontinuum H of X , the condition $H \cap G = \emptyset$ implies that H is tree-like.*

Since the arc component A is dense in X , it follows from Proposition 2.3 that there is a sequence of mappings $f_n : X \rightarrow X$ such that $G \cap f_n(H) = \emptyset \neq A \cap f_n(H)$ and $\lim d(f_n) = 0$. Applying Claim 1, observe that $f_n(H) \subset X \setminus \text{Ker}(X)$; thus, $f_n(H)$ are tree-like. Since H has ε -mappings onto tree-like continua for each $\varepsilon > 0$, H is tree-like. So, Claim 2A follows.

Denote by C the composant of the kernel $\text{Ker}(X)$ containing the continuum G .

CLAIM 3A. *For each subcontinuum H of X , the condition $H \cap \text{Ker}(X) \subset C$ implies that H is tree-like.*

Put $H_0 = H \cap \text{Ker}(X)$. If $H_0 = \emptyset$, then H is a dendroid, hence, the conclusion follows. So assume that $H_0 \neq \emptyset$. Using the density of $\text{Ker}(X) \setminus C$ in $\text{Ker}(X)$ and applying (2.2), we find a sequence of mappings $f_n : X \rightarrow X$, each moving a point of H_0 to a point of $\text{Ker}(X) \setminus C$, such that $\lim d(f_n) = 0$. Then, for almost all $n \in \mathbb{N}$, we have

$$(\text{Ker}(X) \setminus C) \cap f_n(H) \neq \emptyset, \text{ and}$$

$$(X \setminus \text{Ker}(X)) \cap f_n(H) \neq \emptyset \neq \text{Ker}(X) \setminus f_n(H).$$

The sets $\text{Ker}(X) \cap f_n(H)$ are connected for each n . Since $\text{Ker}(X)$ is irreducible between any pair of points in C and $\text{Ker}(X) \setminus C$, respectively, it follows that $G \cap f_n(H) \subset C \cap f_n(H) = \emptyset$. Thus, the continua $f_n(H)$ are tree-like according to Claim 2A. So the continuum H has ε -mappings onto tree-like continua for each $\varepsilon > 0$; therefore, it is tree-like itself. Claim 3A is shown.

Since X is hereditarily unicoherent and $\text{Ker}(X)$ is indecomposable, it follows that for each subcontinuum H of X that does not contain $\text{Ker}(X)$, the intersection $H \cap \text{Ker}(X)$ is contained either in the composant C or in $\text{Ker}(X) \setminus C$. Using Claims 2A and 3A, we infer that the conclusion of the theorem holds in Case A.

Case B. $L = \emptyset$ and $K_0 = \text{Ker}(X)$.

If H is a subcontinuum of X that does not contain $\text{Ker}(X)$, then using again Proposition 2.3, we can choose a sequence of mappings $f_n : X \rightarrow X$ such that $A \cap f_n(H) \neq \emptyset \neq \text{Ker}(X) \setminus f_n(H)$ and $\lim d(f_n) = 0$. Using the irreducibility of K and the hereditary unicoherence of X one can show that if $K_0 = \text{Ker}(X)$ is not contained in a continuum Z intersecting A , then $Z \cap \text{Ker}(X) = \emptyset$. Therefore, $\text{Ker}(X) \cap f_n(H) = \emptyset$, so the continua $f_n(H)$ are tree-like. As previously, we infer that H is tree-like. So the conclusion holds also in Case B.

The proof is complete. \square

Corollary 4.10. *If $X \in \text{AR}(\mathcal{HU})$, then either each subcontinuum of X that does not contain $\text{Ker}(X)$ is tree-like, or for each arc component A of X the intersection $A \cap \text{Ker}(X)$ is a dense arc component of $\text{Ker}(X)$.*

Corollary 4.11. *If $X \in \text{AR}(\mathcal{HU})$ and $\dim X > 1$, then for each arc component A of X the intersection $A \cap \text{Ker}(X)$ is a dense arc component of $\text{Ker}(X)$.*

We end the paper with the following question which is the most important one in the area studied in this paper.

Question 4.12. Is every member of $\text{AR}(\mathcal{HU})$ a tree-like continuum?

REFERENCES

- [1] D. P. Bellamy, *An interesting plane dendroid*, Fund. Math. **110** (1980), 191–208.
- [2] K. Borsuk, *Theory of Retracts*, PWN, Warszawa, 1967.
- [3] J. J. Charatonik, *On ramification points in the classical sense*, Fund. Math. **51** (1962), 229–252.
- [4] J. J. Charatonik, W. J. Charatonik and J. R. Prajs, *Arc property of Kelley and absolute retracts for hereditarily unicoherent continua*, (preprint).
- [5] J. J. Charatonik and C. Eberhart, *On smooth dendroids*, Fund. Math. **67** (1970), 297–322.
- [6] W. J. Charatonik, *Arc approximation property and confluence of induced mappings*, Rocky Mountain J. Math. **28** (1998), 107–154.
- [7] S. T. Czuba, *On dendroids with Kelley's property*, Proc. Amer. Math. Soc. **102** (1988), 728–731.
- [8] J. Dydak and J. Segal, *Shape Theory*, Lecture Notes in Math. vol. 688 (eds. A. Dold and B. Eckmann), Springer-Verlag, Berlin, Heidelberg and New York, 1978.
- [9] S. T. Hu, *Theory of Retracts*, Wayne State University Press, Detroit, 1965.

- [10] J. Krasinkiewicz, *Curves which are continuous images of tree-like continua are movable*, *Fund. Math.* **89** (1975), 233–263.
- [11] K. Kuratowski, *Topology, Vol. 1*, Academic Press and PWN, New York, London and Warszawa, 1966.
- [12] K. Kuratowski, *Topology, Vol. 2*, Academic Press and PWN, New York, London and Warszawa, 1968.
- [13] T. Maćkowiak, *Continuous mappings on continua*, *Dissertationes Math. (Rozprawy Mat.)* **158** (1979), 1–95.
- [14] T. Maćkowiak, *Retracts of hereditarily unicoherent continua*, *Bull. Acad. Polon. Sci. Ser. Sci. Math.* **28** (1980), 177–183.
- [15] S. B. Nadler, Jr., *Multicoherence techniques applied to inverse limits*, *Trans. Amer. Math. Soc.* **157** (1971), 227–234.
- [16] S. B. Nadler, Jr., *Hyperspaces of Sets*, M. Dekker, New York and Basel, 1978.
- [17] R. W. Wardle, *On a property of Kelley*, *Houston J. Math.* **3** (1977), 291–299.
- [18] G. T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Colloq. Publ. 28, Providence, 1942 (reprinted with corrections 1971).

(J. Charatonik) MATHEMATICAL INSTITUTE, UNIVERSITY OF WROCLAW, PL.
 GRUNWALDZKI 2/4, 50-384 WROCLAW, POLAND
 INSTITUTO DE MATEMÁTICAS, UNAM, CIRCUITO EXTERIOR, CIUDAD UNIVER-
 SITARIA, 04510 MÉXICO, D. F., MÉXICO
E-mail address: `jjc@matem.unam.mx`

(W. Charatonik) DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVER-
 SITY OF MISSOURI-ROLLA, ROLLA, MO 65409-0020 USA
E-mail address: `wjcharat@umr.edu`

(Prajs) INSTITUTE OF MATHEMATICS, UNIVERSITY OF OPOLE, UL. OLESKA
 48, 45-052 OPOLE, POLAND
 DEPARTMENT OF MATHEMATICS, IDAHO STATE UNIVERSITY, POCA TELLO,
 ID 83209 USA
E-mail address: `prajs@isu.edu`