ON THE PROPERTY OF KELLEY IN HYPERSONES

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Nadler has shown ([3], (16.35), p. 558) an example of a continuum X having the property of Kelley and such that \( X \times X \) fails to have the property. Next he asks some questions on connections between the property of Kelley for X and for the hyperspaces \( C(X) \) and \( 2^X \) ([3], (16.37), p. 558). Two of them are: If X has the property of Kelley, then does \( 2^X \) have it, too? If \( C(X) \) has the property of Kelley, then does \( 2^X \) have it, too? The example mentioned above is such that X has the property of Kelley, \( C(X) \) has it, but \( 2^X \) does not have the property. The present paper recalls the construction of the example and shows ideas of proofs of these facts. The complete proofs are presented in [1].

The statements that the property of Kelley for the hyperspaces \( 2^X \) or \( C(X) \) implies the property for X are shown in [5], (2.8), p. 294 (it is shown for \( 2^X \) only, but the proof is also applicable for \( C(X) \)).

Thus two questions only are open in this area: 1° Is it true that if X has the property of Kelley, then \( C(X) \) has also this property? 2° Is it true that if \( 2^X \) has the property of Kelley, then \( C(X) \) has it? Note that an affirmative answer to 1° implies the same for 2°. One can ask some other questions, that also concern the property of Kelley for hyperspaces, but which are related to Whitney properties. Three of them are:

Is the property of Kelley a Whitney property? Is it a (strong) Whitney reversible property? The present paper contains a partial answer to the first question only. Namely it is shown that the property of Kelley is a Whitney property for continua having covering property hereditarily.

All spaces considered in this paper are assumed to be metric continua. For a given space X with a metric d let \( 2^X \) and \( C(X) \) be the hyperspaces of all compact subsets and of all subcontinua of X respectively with the Hausdorff distance dist defined by

\[
\text{dist} (A,B) = \inf \{ \varepsilon > 0 : A \subset N(B,\varepsilon) \text{ and } B \subset N(A,\varepsilon) \},
\]
where $N(A,\varepsilon)$ is the union of the $\varepsilon$-balls about the points of $A$. We shall also be considering the hyperspace $C^2(X) = C(C(X))$ with the Hausdorff distance denoted $\text{Dist}$.

We say that a continuum $X$ has the property of Kelley if it satisfies the condition: given $\varepsilon > 0$, there exists $\delta > 0$ such that if $a, b \in X$ with $d(a, b) < \delta$ and $a \in A \subset C(X)$, then there is a continuum $B$ with $b \in B$ such that $\text{dist}(A, B) < \varepsilon$.

A continuous mapping $\mu: C(X) \rightarrow [0, \infty)$ is called a Whitney map if $\mu(\{x\}) = 0$ for $x \in X$ and $\mu(A) < \mu(B)$ for $A \subset B$. A topological property $P$ is called a Whitney property ([3], p. 399) if for each continuum $X$ and each Whitney map $\mu$, the implication $X \in P \Rightarrow \mu^{-1}(t) \in P$ holds for each $t \in [0, \mu(X)]$. A topological property $P$ is called a (strong) Whitney reversible property ([3], (14.45), p. 453) provided that whenever $X$ is a continuum such that $\mu^{-1}(t)$ has property $P$ for (some Whitney map $\mu$) all Whitney maps $\mu$ of $C(X)$ and all $0 < t < \mu(X)$, then $X$ has property $P$.

A continuum $X$ is said to have the covering property ([3], (14.14) p. 417) if, for each Whitney map $\mu$, no proper subcontinuum of $\mu^{-1}(t)$ for any $t \in [0, \mu(X)]$ covers $X$. If each subcontinuum of $X$ has the covering property, then we say that $X$ has the covering property hereditarily ([3], p. 486). The statement that a continuum $X$ has the covering property hereditarily is equivalent to the fact that for all $\mathcal{A} \in C(\mu^{-1}(t))$, with $t \in [0, \mu(X)]$, we have $\mathcal{A} = C(\cup \mathcal{A}) \cap \mu^{-1}(t) = \{X \in C(X) \mid K \subset \cup \mathcal{A} \text{ and } \mu(K) = t\}$ (see [4], p. 159).

The reader is referred to [2] for definitions of terms not given here.

Let $R$ denote the real line $(-\infty, \infty)$ and let $H$ be the half line $[1, \infty)$. We show the following

EXAMPLE. There exists a continuum $X$ having the property of Kelley and such that the hyperspace $C(X)$ has, while $2^X$ does not have this property.

Define the mappings $g$ and $f$ from $H$ into $R^2$ by $g(t) = (1 - 1/t) \exp(-it)$ and $f(t) = (1 + 1/t) \exp(it)$, and let $L = g(H)$ and $M = f(H)$. Denote by $S$ the usual unit circle in $R^2$. The space $X = L \cup S \cup M$ is a continuum in $R^2$. It can be observed that $X$ has the property of Kelley. Using the fact that the hyperspaces $C(\cup S)$ and $C(M \cup S)$ are homeomorphic to cones over these continua, one can prove that the hyperspace $C(X)$ has the property of Kelley.

Now assume $2^X$ has the property of Kelley and take $0 < \varepsilon < 1/2$. By assumption there exists a number $\delta$ with $0 < \delta < \varepsilon$ satisfying the definit
of the property. Let \( \mathcal{F} \) be the set of all singletons in \( X \). Since \( \mathcal{F} \) is homeomorphic to \( X \), it is a subcontinuum of \( \mathfrak{c}(X) \). Let \( p \in \mathcal{L} \) and \( q \in \mathcal{M} \) be two points such that \( d(p,q) < \delta \). Then \( \text{dist}(\{p,q\}, \{p\}) = d(p,q) < \delta \) and, by assumption, there exists a continuum \( \mathcal{K} \subset \mathfrak{c}(X) \) containing \( \{p,q\} \) with \( \text{Dist}(\mathcal{K}, \mathcal{F}) < \epsilon \). In particular, elements of \( \mathcal{K} \) have diameters less than \( 2\epsilon < 1 \). Taking an element \( A \in \mathcal{K} \) satisfying \( \text{dist}(A, \{g(1)\}) < \epsilon \) we have \( A \subset \mathcal{L} \). Since the spirals \( L \) and \( M \) wind up the circle \( S \) in opposite directions, we cannot join \( A \) and \( \{p,q\} \) continuously by a family of sets of small diameters. This contradicts to the existence of the continuum \( \mathcal{K} \).

**Theorem.** The property of Kelley is a Whitney property in the class of continua having covering property hereditarily.

To prove the theorem we need some auxiliary concepts and two lemmas. Fix a continuum \( X \) and let \( \mu \) be a Whitney map for \( C(X) \). Define \( \mathfrak{F}_\mu : X \times [0, \mu(X)] \to C^2(X) \) by \( \mathfrak{F}_\mu(a,t) = \{x \in C(X): a \in x \in \mu^{-1}(t)\} \) (see [3], (16.12), p. 539), and \( \mathfrak{F}_\mu^*: C(X) \times [0, \mu(X)] \to C^2(X) \) by

\[
\mathfrak{F}_\mu^*(A,t) = \begin{cases} 
\{x \in C(X): A \subset x \in \mu^{-1}(t)\}, & \text{if } \mu(A) \leq t, \\
\{A\}, & \text{if } \mu(A) > t.
\end{cases}
\]

It is proved in [3], (16.14), p. 539 that

**Lemma 1.** Continuity of \( \mathfrak{F}_\mu \) is equivalent to the property of Kelley for \( X \).

We need a stronger result. Namely we prove that

**Lemma 2.** Continuity of \( \mathfrak{F}_\mu^* \) is equivalent to the property of Kelley for \( X \).

**Proof.** Assume first \( \mathfrak{F}_\mu^* \) is continuous. Then it is continuous at each point \( (a,t) \) and thus \( \mathfrak{F}_\mu \) is continuous at each \( (a,t) \). So \( X \) has the property of Kelley by Lemma 1.

We claim that

(\#) for all continua \( X \) the function \( \mathfrak{F}_\mu^* \) is upper semi-continuous.

Take arbitrary convergent sequences \( A_n \to A \) and \( t_n \to t \). We show \( Ls \mathfrak{F}_\mu^*(A_n, t_n) \subset \mathfrak{F}_\mu^*(A, t) \). The case \( \mu(A) \geq t \) is trivial, so we can assume \( \mu(A) < t \) and \( \mu(A_n) < t_n \). Let \( K \in Ls \mathfrak{F}_\mu^*(A_n, t_n) \). Hence there is a subsequence \( K_{n_m} \in \mathfrak{F}_\mu^*(A_{n_m}, t_{n_m}) \) tending to \( K \). Thus applying the definition of \( \mathfrak{F}_\mu^* \) and taking the limit we get \( A \subset K \) and \( \mu(K) = t \), i.e., \( K \in \mathfrak{F}_\mu^*(A, t) \). So (\#) is true.

Assume now \( X \) has the property of Kelley. To prove continuity of
it is enough to show, by (*), its lower semi-continuity. To this end consider arbitrary convergent sequences $A_n \to A$ and $t_n \to t$. We have to show $E_\mu(A,t) \subseteq \text{Li}_\mu(E_\mu(A,t_n))$. Note that the case $\mu(A) \geq t$ is trivial again, so we can assume $\mu(A) < t$ and $\mu(A_n) < t_n$. Take $K \in E_\mu(A,t)$, i.e., $A \subseteq K \subseteq C(X)$ and $\mu(K) = t$. For each $n \in \{1,2,\ldots\}$ choose a point $a_n \in A_n$ such that $a_n \to a \in A$. Since $E_\mu$ is continuous by Lemma 1, there is a sequence of continua $K'_n$ with $a_n \in K'_n$ and $\mu(K'_n) = t_n$. Let $K_n$ be such a continuum in $X$ that $A_n \subseteq K_n \subseteq A_n \cup K'_n$ and $\mu(K_n) = t_n$. Then $K_n \in E_\mu(A_n,t_n)$. We shall show $K_n \to K$. Observe that $\text{Li}_\mu(K_n) \subseteq \text{Li}_\mu(K)$ and $\text{Li}_\mu(K)$ is defined by $\text{Li}_\mu(K) = \{t \in [0,\infty) : \mu(A,t) = \mu(K) = t\}$, so $\mu(L) < \mu(K)$ by the definition of $\mu$, but $\mu(L) = \lim \mu(K'_n) = \lim t_n = t = \mu(K)$, a contradiction. Therefore $E_\mu$ is continuous, and the proof is complete.

**Proof of the theorem.** Let $X$ be a continuum having the property of Kelley and covering property hereditarily, and let $\mu$ be an arbitrary Whitney map for $C(X)$. Fix $t_0 \in [0,\mu(X)]$. We have to show that $\mu^{-1}(t_0)$ has the property of Kelley. For this purpose note that $C(\mu^{-1}(t_0))$ is homeomorphic to $\mu^{-1}([t_0,\mu(X)])$ and the homeomorphism $\sigma$ is defined by $\sigma(\mathcal{R}) = \mathcal{R}$ (see [4], Corollary 16, p. 161). Define a mapping $M : C(\mu^{-1}(t_0)) \to [0,\infty)$ by $M(\mathcal{R}) = \mu(\sigma(\mathcal{R})) - t_0$. It is shown in [4], Lemma 25, p. 164 that $M$ is a Whitney map. Now, the property of Kelley for $\mu^{-1}(t_0)$ is equivalent, by Lemma 1, to continuity of the function $F_M : \mu^{-1}(t_0) \times [0,\mu(\mu^{-1}(t_0))]) \to C(\mu^{-1}(t_0))$. Since $F_M(A,t) = \{K \in C(\mu^{-1}(t_0)) : A \subseteq K \} = \sigma^{-1}(F_\mu(A,t + t_0))$. Thus continuity of $F_M$ follows from one of $F_\mu$ (see Lemma 2) and of $\sigma^{-1}$. Hence the proof is complete.

**References**