On Mazurkiewicz sets

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Abstract. A Mazurkiewicz set \( M \) is a subset of a plane with the property that each straight line intersects \( M \) in exactly two points. We modify the original construction to obtain a Mazurkiewicz set which does not contain vertices of an equilateral triangle or a square. This answers some questions by L.D. Loveland and S.M. Loveland. We also use similar methods to construct a bounded noncompact, nonconnected generalized Mazurkiewicz set.

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By a Mazurkiewicz set (shortly M-set) we mean a subset \( X \) of the plane such that every straight line intersects \( X \) in exactly two points. It was constructed in [3] using transfinite induction. The notion was generalized in two directions: to generalized Mazurkiewicz sets (GM-sets) and to sets with the double midset property (DMP-sets). Let us recall that a subset \( X \) of the plane is a GM-set if it contains at least two points and each line that separates two points of \( X \) intersects \( X \) in exactly two points. A subset \( X \) of the plane is a DMP-set if it contains at least two points and the perpendicular bisector of every segment joining two points in \( X \) intersect \( X \) in exactly two points. It follows from the definitions that every M-set is a GM-set and every GM-set is an DMP-set. For more information about these notions see [2]. In the same article the authors ask some questions related to the subject. Here we answer some of them in a more general case constructing, using transfinite induction, an M-set with some additional geometrical properties. Namely, the M-set that does not contain vertices of an equilateral triangle or vertices of a square, and whose image under the inversion with respect to the unit circle is a bounded, noncompact, nonconnected GM-set.

We will need some denotation. The symbol \( \mathfrak{c} \) denotes the cardinal number continuum, i.e. the first ordinal number whose cardinality is the cardinality of reals. All the constructions are going to be done in the complex plane \( \mathbb{C} \). Given \( x, y \in \mathbb{C} \) the symbol \( l(x, y) \) denotes the line through \( x \) and \( y \) if \( x \neq y \), and \( l(x, x) = \{x\} \). If both \( x \) and \( y \) are distinct from 0, and \( x \neq y \), then \( c(x, y) \) is the circle that contains \( x, y \) and 0. Moreover we put \( c(x, x) = \{x\} \). For a subset \( A \subset \mathbb{C} \) we put \( L(A) = \bigcup \{l(x, y) : x, y \in A\} \) and \( C(A) = \bigcup \{c(x, y) : x, y \in A\} \).

We denote by \( B \) the open unit disk in the plane, i.e. \( B = \{z \in \mathbb{C} : |z| < 1\} \).
Theorem. There is an $M$-set $A$ satisfying the following conditions:

1. $A$ does not contain vertices of an equilateral triangle;
2. $A$ does not contain vertices of a right isosceles triangle;
3. $A \cap \text{cl } B = \emptyset$;
4. any circle that contains 0 and is not contained in $\text{cl } B$ intersects $A$ at exactly two points.

Proof: Given two different points $a, b \in \mathbb{C}$ define $P(a, b)$ as the set of all points $x \in \mathbb{C}$ such that the triangle with vertices $a, b, x$ is an equilateral one or a right isosceles one. Thus $P(a, b)$ has exactly eight points. In particular we have $P(0, 1) = \{i, -i, 1 + i, 1 - i, \frac{1}{2} + \frac{\sqrt{2}}{2}i, \frac{1}{2} + \frac{-\sqrt{2}}{2}i, \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} + \frac{-\sqrt{3}}{2}i\}$. Additionally we put $P(x, x) = \emptyset$. For a set $A \subset \mathbb{C}$ let $P(A) = \bigcup\{P(x, y) : x, y \in A\}$.

Let $\{l_\alpha : \alpha \leq \epsilon\}$ be the set of all straight lines in the plane, and let $\{c_\alpha : \alpha \leq \epsilon\}$ be the set of all circles passing through 0 and not contained in $\text{cl } B$. We will define, for $\alpha < \epsilon$, the set $A_\alpha$, and $A = \bigcup\{A_\alpha : \alpha < \epsilon\}$ will be the required $M$-set. Assume that, for some $\alpha < \epsilon$, the sets $A_\beta$ for $\beta < \alpha$, have been defined satisfying the following conditions:

1. $\text{card}(A_\beta) < \epsilon$;
2. for every $\gamma < \beta$ we have $A_\gamma \subset A_\beta$;
3. $A_\beta \cap l_\beta$ is a two point set;
4. $A_\beta \cap c_\beta$ is a two point set;
5. $A_\beta \cap P(A_\beta) = \emptyset$;
6. $A_\beta$ contains no three colinear points;
7. there is no circle in the plane that contains three different points of $A_\beta$ and the point 0;
8. $A_\beta \cap \text{cl } B = \emptyset$.

Put $N_\alpha = \bigcup\{A_\beta : \beta < \alpha\}$. Then

- $\text{card}(P(N_\alpha)) < \epsilon$,
- $\text{card}(l_\alpha \cap (\bigcup\{l(x, y) : x, y \in N_\alpha\})) < \epsilon$,
- $\text{card}(c_\alpha \cap (\bigcup\{c(x, y) : x, y \in N_\alpha\})) < \epsilon$,
- $\text{card}(l_\alpha \cap N_\alpha) \leq 2$,
- $\text{card}(c_\alpha \cap N_\alpha) \leq 2$.

Thus we can choose points $x_\alpha, y_\alpha, z_\alpha, t_\alpha$ that satisfy the following conditions, where $G_\alpha = \text{cl } B \cup P(N_\alpha) \cup L(N_\alpha) \cup C(N_\alpha)$.

- $x_\alpha, y_\alpha \in l_\alpha \setminus c_\alpha$,
- $z_\alpha, t_\alpha \in c_\alpha \setminus l_\alpha$,
- if $\text{card}(l_\alpha \cap N_\alpha) = 2$, then $\{x_\alpha, y_\alpha\} = l_\alpha \cap N_\alpha$,
- if $\text{card}(c_\alpha \cap N_\alpha) = 2$, then $\{z_\alpha, t_\alpha\} = c_\alpha \cap N_\alpha$,
- if $\text{card}(l_\alpha \cap N_\alpha) = 1$, then $\{x_\alpha\} = l_\alpha \cap N_\alpha$ and $y_\alpha \notin G_\alpha$,
- if $\text{card}(c_\alpha \cap N_\alpha) = 1$, then $\{z_\alpha\} = c_\alpha \cap N_\alpha$ and $t_\alpha \notin G_\alpha$,
- if $\text{card}(l_\alpha \cap N_\alpha) = 0$, then $x_\alpha, y_\alpha \notin G_\alpha$,
- if $\text{card}(c_\alpha \cap N_\alpha) = 0$, then $z_\alpha, t_\alpha \notin G_\alpha$. 
Finally put $A_{\alpha} = N_{\alpha} \cup \{x_{\alpha}, y_{\alpha}, z_{\alpha}, t_{\alpha}\}$. One can verify that, by the construction, conditions (1$\alpha$)–(8$\alpha$) are satisfied. Putting $A = \bigcup\{A_{\alpha} : \alpha < c\}$ we see that $A$ is the required M-set. This finishes the proof.

**Remark 1.** In [2, Questions 2 and 3, p.488] the authors asked if there is a DMP-set in the plane that does not contain vertices of a square (Question 2) and if there is a DMP-set in the plane that does not contain vertices of an equilateral triangle (Question 3). Because every M-set is a DMP-set, the Theorem answers both questions.

Denote by $h : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \setminus \{0\}$ the inversion with respect to the unit circle, i.e. $h(z) = 1/\overline{z}$. Observe that $h(h(z)) = z$.

**Proposition.** Let $A$ be an M-set that satisfies conditions (3) and (4) of the Theorem. Then $h(A)$ is a GM-set.

**Proof.** First observe that $h(A) \subset B$ by condition (3). Let $l$ be a line that separates two points of $h(A)$. If $0 \in l$, then $h(l \setminus \{0\}) = l \setminus \{0\}$. If $0 \notin l$, then $h(l)$ is a circle passing through 0 and not contained in $B$. In any case $h(l) \cap A$ is a two point set by (4), and therefore $h(h(l)) \cap h(A) = l \cap h(A)$ is a two point set, as required.

**Remark 2.** In [2, Question 6, p.490] the authors ask the following question. Is there a bounded GM-set which is not a simple closed curve? Is a bounded GM-set necessarily closed? Connected? Since the constructed set $h(A)$ is a bounded GM-set homeomorphic to an M-set, it is neither closed (M-sets are not bounded, so not compact) nor connected (M-sets are zerodimensional, see [1, Theorem 2, p.553]). Thus the Proposition answers in the negative all of the three questions. It also answers more particular Question 7 and partially Question 8.

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