MEASURES ON HYPERSPACES

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Abstract. From a measure space, \((X, \mu_X)\) we define a measure \(\mu^{P(X)}\) on the power set of \(X\). If \((X, \tau)\) is a compactum, whose topology \(\tau\) is compatible with the measure \(\mu_X\) on \(X\), then the measure \(\mu^{P(X)}\) restricts to a natural measure on the hyperspace of closed sets of that given compactum. Surprisingly, under very mild conditions, \(\mu^{P(X)}\) is always supported on the hyperspace of finite subsets.

1. Introduction and preliminaries

1.1. The problem. One of the main difficulties encountered in hyperspace theory is the problem of “lifting” structure from the underlying space up to its hyperspaces. The Vietoris [13] and Fell [5] topologies, and some others, provide ways to do this for topological structure, so that important topological properties, like compactness, connectedness, etc., are preserved. The Hausdorff metric, and several other metrics (see [2], [3], and [4]), provide for a “lifting” of metrics to hyperspaces that is compatible with these topologies. Here, we will deal with the corresponding measure extension problem that arises when a given space carries a measure.

For more information about continua and their hyperspaces, see [10] or [6].
For more information about measures and measure theory, see the classical works [1], [7], [8], [9], or the more modern one [12].

2. Construction of the measure

The following definition is inspired by the definition of basis elements for the Vietoris topology; however, it is not restricted to open sets in a base topological space.

Definition 2.1. Let \(X\) be a set, and let \(A_1, \ldots, A_n \subseteq X\). Then
\[
\langle A_1, \ldots, A_n \rangle = \{ P \subseteq A_1 \cup \cdots \cup A_n | j \in \{1, \ldots, n\} \Rightarrow P \cap A_j \neq \emptyset \}.
\]
Let \(X = (X, \mathcal{M}^X, \mu^X)\) be a measure space. That is, \(\mathcal{M}^X\) is a \(\sigma\)-subalgebra of the power set of \(X\) that supports a given measure, \(\mu^X\), on the set \(X\). Then
\[
\mathcal{P}(X) = \left( \mathcal{P}(X), \mathcal{M}^{P(X)}, \mu^{P(X)} \right),
\]
where
\[
(1) \ \mathcal{P}(X) \text{ is the power set of } X: \mathcal{P}(X) = \{ A | A \subseteq X \}.
\]
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(2) $\mu^{P(X)}$ is the measure defined by the formula

$$\mu^{P(X)}(\langle A_1, \ldots, A_n \rangle) = \left( 2^{\mu^X(A_1)} - 1 \right) \cdots \left( 2^{\mu^X(A_n)} - 1 \right),$$

for any pairwise disjoint finite list $A_1, \ldots, A_n$, each member of which is a $\mu^X$-measurable set, such that either all are of finite $\mu^X$ measure or all are of positive $\mu^X$ measure.

(3) $\mathcal{M}^{P(X)}$ is the set (i.e. the $\sigma$-algebra of all $\mu^{P(X)}$-measurable subsets of $\mathcal{P}(X)$. That is, a subset $P$ of $\mathcal{P}(X)$ is a member of the set $\mathcal{M}^{P(X)}$ precisely if the inner and outer measures determined by $\mu^{P(X)}$, namely $\mu^{P(X)}_*$ and $\mu^{P(X)}^*$, agree on $P$:

$$\mu^{P(X)}_*(P) = \mu^{P(X)}^*(P).$$

Remark 2.2. Definition 2.1 yields counting measure in the case that the set $X$ is finite. In fact, this was our motivation for crafting the formula for this extension of the measure.

Let us call a subset of $\mathcal{P}(X)$ a basic $\mu^{P(X)}$-measurable set if it is of the form $\langle A_1, \ldots, A_n \rangle$, where $A_1, \ldots, A_n$ are $\mu$-measurable subsets of $X$.

To verify that we actually have defined a measure on $\mathcal{P}(X)$, we need to check that it does not lead to a contradiction, i.e., if $A, A_1, \ldots, A_n$ are basic $\mu^{P(X)}$-measurable sets, $A = A_1 \cup \cdots \cup A_n$, and $A_i \cap A_j = \emptyset$ for $i \neq j$, then $\mu^{P(X)}(A) = \sum_1^n A_j$.

The simplest such case is presented in a theorem below. The remaining cases follow from this one by induction.

2.1. Preliminary observations.

Theorem 2.3. Let $X$ be a set, and let $A, B, C, \ldots, C_n$ be pairwise disjoint subsets of $X$. Then

$$\langle A \cup B, C_1, \ldots, C_n \rangle = \langle A, C_1, \ldots, C_n \rangle$$

$$\cup \langle B, C_1, \ldots, C_n \rangle \cup \langle A, B, C_1, \ldots, C_n \rangle,$$

and

$$\mu^{P(X)}(\langle A \cup B, C_1, \ldots, C_n \rangle) = \mu^{P(X)}(\langle A, C_1, \ldots, C_n \rangle)$$

$$+ \mu^{P(X)}(\langle B, C_1, \ldots, C_n \rangle) + \mu^{P(X)}(\langle A, B, C_1, \ldots, C_n \rangle).$$

Proof. The first equation is an easy set-theoretic consequence of our definitions and notation. To see that the second equation holds, set $\alpha = 2^{\mu^X(A)}$, $\beta = 2^{\mu^X(B)}$, and $\gamma = 2^{\mu^X(C)}$. Then we have

$$\mu^{P(X)}(\langle A \cup B, C_1, \ldots, C_n \rangle)$$

$$= (\alpha \beta - 1) \gamma$$

$$= (\alpha - 1) \gamma + (\beta - 1) \gamma + (\alpha - 1)(\beta - 1) \gamma$$

$$= \mu^{P(X)}(\langle A, C_1, \ldots, C_n \rangle) + \mu^{P(X)}(\langle B, C_1, \ldots, C_n \rangle)$$

$$+ \mu^{P(X)}(\langle A, B, C_1, \ldots, C_n \rangle),$$

as desired. \qed

Remark 2.4. If $\mu^X(X)$ is infinite, then so is $\mu^{P(X)}(\{X\})$. 

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Definition 2.5. A set $A$ in $M^X$ is called an atom if $\mu^X(A) > 0$ and for any measurable subset $B$ of $A$ with $\mu^X(B) < \mu^X(A)$ one has $\mu^X(B) = 0$. A measure which has no atoms is called non-atomic.

The following relevant result is well known, and is due to Sierpiński [11].

Theorem 2.6. Let $(X, \mu^X)$ be a non-atomic measure space, and let $\varepsilon \leq \mu^X(X)$ be positive and finite. If $p \in X$, then there is a $\mu^X$-measurable set $E \subseteq X$ such that $p \in E$ and $\mu^X(E) = \varepsilon$.

As an immediate consequence, we get the following helpful result.

Lemma 2.7. Let the measure space $(X, \mu^X)$ be non-atomic. If $\mu^X(X) = 1$, then $X$ has a family, $C = \{C_m|1 \leq m < \infty\}$, of coverings by disjoint non-empty measurable subsets, such that for each positive integer $m$, $A \in C_m$ implies that $\mu^X(A) = \frac{1}{m}$.

Notation 2.8. Let $C = \{C_m|1 \leq m < \infty\}$ be a family of coverings of $X$ by disjoint non-empty subsets. For each pair, $m, k$, of positive integers,

$$F_{m,k}(C) = \bigcup \{\langle U_1, \ldots, U_k \rangle | U_1, \ldots, U_k \in C_m \text{ and } |\{U_1, \ldots, U_k\}| = k\}.$$

In other words, $F_{m,k}(C)$ is the family of all sets that can be covered by exactly $k$ elements of $C_m$.

Lemma 2.9. Let the measure space $(X, \mu^X)$ be non-atomic with $\mu^X(X) = 1$, and let $C = \{C_m|1 \leq m < \infty\}$ be a family of coverings of $X$ by disjoint non-empty subsets, such that, for each positive integer $m$, $A \in C_m$ implies that $\mu^X(A) = \frac{1}{m}$.

Given a positive integer $k$, the collection

$$F_k(C) = \bigcap \left\{\bigcup \{F_{m,k}(C) | M \leq m < \infty\} | 1 \leq M < \infty\right\}$$

consists of sets of measure zero. Moreover, we have

$$\mu^{P(X)}(F_k(C)) = \frac{1}{k!} (\ln(2))^k.$$

Proof. Let $A$ be an element of $F_k(C)$. Then, for any $M > 0$, there is $m \geq M$ such that $A$ can be covered by exactly $k$ elements of $C_m$. That each member of $F_k(C)$ has measure zero follows. Finally,

$$\mu^{P(X)}(F_k(C)) = \lim_{m \to \infty} \left[ \binom{m}{k} \left( \frac{2^m}{2^m} - 1 \right)^k \right] = \lim_{m \to \infty} \left[ \frac{1}{k!} \left( \frac{2^m}{2^m} - 1 \right)^k \right] = \frac{1}{k!} (\ln(2))^k,$$

as required. \qed

Remark 2.10. If $X = [0,1]^n$ with Lebesgue measure, then we may choose the coverings in the family $C$ so that the diameters of the elements of $C_k$ tend to zero as $k \to \infty$. Then the collection $F_k(C)$ is the hyperspace of all $k$-element subsets of $X$.

Theorem 2.11 (Main theorem). Let the measure space $(X, \mu^X)$ be non-atomic with $\mu^X$ a finite measure. Then the measure $\mu^{P(X)}$ is supported on the hyperspace $F(X) = \{A \subseteq X|\mu^X(A) = 0\}$. In other words, in hyperspaces, under these hypotheses, a typical set is a set of measure zero.
Proof. For simplicity, we assume, without loss of generality, that the measure on $X$ is normalized, i.e., that $\mu^X(X) = 1$. Since $F(X) \supseteq \bigcup_{k=1}^{\infty} (F_k(C))$, we have that
\[
\mu^{P(X)}(F(X)) \geq \sum_{k=1}^{\infty} \left[ \mu^{P(X)}(F_k(C)) \right] = \sum_{k=1}^{\infty} \left[ \frac{1}{k!} (\ln(2))^k \right] = 1,
\]
the measure $\mu^{P(X)}$ is supported on the hyperspace $F(X) = \{A \subseteq X|\mu^X(A) = 0\}$, as claimed. \hfill \Box

3. Connections to topology

The proof of the following theorem is immediate from the definitions.

**Theorem 3.1.** The measure $\mu^{P(X)}$ has the following properties, when the space $X$ carries a given topology, $\tau$, with respect to which each Borel set is $\mu^X$-measurable:

1. The set $2^X$ of closed subsets of $(X, \tau)$ has full measure, so that restriction of $\mu^{P(X)}$ to the measurable subsets of the hyperspace $2^X$ yields a measure.
2. If the original measure has the property that every non-empty open set has positive measure, then the same holds for $2^X$, with the Vietoris or Fell topology.

Consequently, we have the following interesting corollary.

**Corollary 3.2.** In the hyperspace of subcontinua of a continuum, a typical set is degenerate.

**References**


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