GLASNIK MATEMATIČKI
Vol. 20 (40) (1985), 139—151.

MONOTONEITY RELATIVE TO A POINT AND INVERSE LIMITS OF CONTINUA

J. J. Charatonik and **W. J. Charatonik,** Wrocław, Poland

Abstract. It is shown that the inverse limit of an inverse system of smooth continua is a smooth continuum provided that there exists a thread composed of initial points and that the bonding mappings are monotone relative to these points. It also is proved that the property of being an arboroid, a dendroid, a generalized tree, a fan or a smooth fan is an invariant under the inverse limit operation if the above assumptions are satisfied for the inverse systems.

The aim of this paper is to investigate some properties of mappings which are monotone relative to a point [18]. After preliminaries, in § 2 we prove some necessary and sufficient conditions under which a mapping $f : X \to Y$ is monotone relative to a point $p \in X$. The conditions are formulated in terms of some quasi-orders on X and on Y, and are applied in the further results. The rest of the paper is devoted to the inverse limits. In § 3 it is shown that smoothness of continua (in the sense of Gordh [10]) is preserved under the inverse limit operation if the bonding mappings are monotone relative to points which form a thread, and that the property of being an arboroid (a dendroid, a generalized tree, a fan, a smooth fan) is also an invariant of this operation, while the property of being a dendrite is not (§ 4). The fourth paragraph contains some examples showing that the assumptions concerning the inverse systems are essential in the theorems proved in § 3. Further, some problems, in particular related to contractibility of continua and to the existence of a continuous selection on the family $C(X)$ of subcontinua of a given continuum X are asked.

The authors thank the referee for pointing out some inaccuracies in an earlier version of this article.

§ 1. Preliminaries

Topological spaces considered throughout this paper are assumed to be compact (thus Hausdorff, see [9], p. 165) and the mappings

---

Mathematic subject classifications (1980): Primary 54C10, 54B25; Secondary 54F20.

Key words and phrases: Dendroid, fan, inverse limit, monotone mapping, quasi-order, smooth, unicoherent.

are assumed to be continuous. By a continuum we mean a compact connected space. An arc is defined as a continuum (not necessarily metrizable) with exactly two non-separating points (called end points of the arc). Note that some authors use, in the same sense, the name of a generalized arc. If p and x are points of a continuum X, then px denotes some subcontinuum of X which is irreducible from p to x. A continuum X is said to be hereditarily unicoherent at a point $p \in X$ if the intersection of any two subcontinua of X, each of which contains p, is connected ([10], p. 52). It is proved ([10], Theorem 1.3, p. 52) that a continuum X is hereditarily unicoherent at a point p if and only if, given any point $x \in X$, there exists a unique subcontinuum px which is irreducible between p and x. A continuum which is hereditarily unicoherent at each point is said to be hereditarily unicoherent.

A mapping $f : X \to Y$ from a topological space X to a topological space Y is called monotone provided that the inverse image of each subcontinuum of Y is a subcontinuum of X (see [26], p. 127 and (2.2), p. 138; cf. [16], § 46, I, Definition and Theorem 9, p. 131). A mapping $f : X \to Y$ of X onto Y is said to be monotone relative to a point $p \in X$ if for each subcontinuum Q of Y such that $f(p) \in Q$ the inverse image $f^{-1}(Q)$ is connected ([18], p. 720). Recall that a mapping is monotone if and only if it is monotone relative to each point ([18], Theorem 2.1, p. 720).

§ 2. Quasi-orders

In the present paragraph we show some necessary and sufficient conditions under which a mapping $f : X \to Y$ is monotone relative to a point $p \in X$. These conditions are expressed in terms of quasi-orders on X and on Y. Before we formulate the main results of the paragraph, we recall the basic concepts and facts we needed.

A quasi-order on a set X is a reflexive and transitive binary relation. If this relation is also antisymmetric, it is called a partial order. Let a quasi-order $\leqslant$ on a set X be given. If $x \leqslant y$ and y non-$\leqslant x$, we write $x < y$. The quasi-order $\leqslant$ is said to be order-dense if whenever $x < y$, there exists $z \in X$ such that $x < z < y$. Given a subset $S \subset X$, an element $z \in S$ is called a zero of X if $z \leqslant x$ for each $x \in S$.

A quasi-order on a topological space X is said to be closed if its graph is a closed subset of the product space $X \times X$. A generalized tree means a hereditarily unicoherent continuum which admits a closed order-dense partial order with a zero (see [24], Theorem 7, p. 801).

If a continuum X is hereditarily unicoherent at a point p, then the quasi-order $\leqslant_p$ on X defined by the condition $x \leqslant_p y$ if and only if $px \subset py$ is said to be a weak cutpoint order with respect to p (see [11], p. 63). Let continua X and Y be hereditarily unicoherent at points p ad q respectively and let $\leqslant_p$ and $\leqslant_q$ be weak cutpoint orders on X and Y with respect to p and q correspondingly. A mapping

$f : X \to Y$ of X onto Y is called order-preserving (or $\leqslant_p$-preserving) if $a \leqslant_p b$ implies $f(a) \leqslant_p f(b)$ for every $a, b \in X$.

The following property of mappings monotone relative to a point will be needed in the sequel.

PROPOSITION 1. *Let a continuum X be hereditarily unicoherent at a point $p \in X$ and let a mapping $f : X \to Y$ be monotone relative to p. If a subcontinuum A of X contains the point p, then the partial mapping $f|A : A \to f(A) \subset Y$ is also monotone relative to p.*

Indeed, take a subcontinuum Q of $f(A)$ such that $f(p) \in Q$, and observe that $(f \mid A)^{-1}(Q) = A \cap f^{-1}(Q)$ is the intersection of two continua both containing the point p, thus it is connected by hereditary unicoherence of X at p.

Note that one cannot replace "monotone relative to p" by "monotone" both in the assumptions and in the conclusion of Proposition 1. Namely let C be an indecomposable continuum and let A be the sin $1/x$-curve, i. e.,

$$A = \{(x, y) \in R^2 : y = \sin 1/x \text{ and } 0 < x \leqslant 1\} \cup \{(0, y) \in R^2 : -1 \leqslant y \leqslant 1\}.$$

Identifying end points of the limit segment of A with two points in distinct composants of C we get a continuum X which is hereditarily unicoherent at the point $p = (1, \sin 1)$ of A. The mapping f on X which shrinks C to a point is monotone, while $f \mid A$ is not.

Now we are ready to formulate and prove the characterizations of monotoneity relative to a point. Some of the equivalences in Proposition 2 below have been established in the metric setting under additional assumptions (see [8], Proposition 4, p. 309; cf. [18], Theorem 2.9 and Corollary 2.10, p. 722).

PROPOSITION 2. *Let continua X and Y be hereditarily unicoherent at points p and q, respectively, and let $f : X \to Y$ be a mapping of X onto Y with $f(p) = q$. Then the following conditions are equivalent:*

(i) f is monotone relative to p,

(ii) $f(px) = f(p) f(x)$ for each $x \in X$,

(iii) f is $\leqslant_p$-preserving,

and every of them is implied by

(iv) $f \mid px$ is monotone for each $x \in X$.

Proof. The equivalence between (*i*) and (*ii*) has been shown in [18], Theorem 2.5, p. 721 and Theorem 2.9, p. 722 (no metric arguments were used in that proof). Assume (*ii*), and let $a \leqslant_p b$ in X. Then $a \in pb$, whence $f(a) \in f(pb) = f(p) f(b)$ by (*ii*), and thus $f(a) \leqslant_q f(b)$, so *(iii)* holds. Now assume *(iii)*, and take a point x in X. Since $f(px)$ is a subcontinuum in Y which contains both $f(p)$ and $f(x)$, and since $f(p)f(x)$ is the only continuum irreducible between these points, we have $f(p)f(x) \subset f(px)$ (cf. [14], Theorem 2—10, p. 44). Invertedly, let $v \in f(px)$. Thus there is a point $u \in px$ with $f(u) = v$. But $u \in px$ means $u \leqslant_p x$, whence $f(u) \leqslant_q f(x)$ by *(iii)*, and therefore $v \in qf(x) = f(p)f(x)$; we have $f(px) \subset f(p)f(x)$, and so *(ii)* is proved. The equivalence between the three conditions is established.

Condition *(iv)* implies *(ii)* because monotone mappings preserve irreducibility of continua (see [16], § 48, I, Theorem 3, p. 192 and note that neither metrizability nor separability of the spaces is used in the proof). So the proposition is proved.

PROPOSITION 3. *Let continua X and Y be hereditarily unicoherent at points p and q, respectively, and let $f : X \to Y$ be a mapping of X onto Y with $f(p) = q$. If X is arcwise connected, or if X is metric and Y is arcwise connected, then all four conditions (i) — (iv) of Proposition 2 are equivalent.*

Proof. By Proposition 2 we need only to show that one of the equivalent conditions *(i)* — *(iii)* implies *(iv)*.

First, let the continuum X be arcwise connected. Then Y is, too (see [23], Theorem, p. 879) and by hereditary unicoherence of Y at q the irreducible continuum $f(p)f(x)$ is an arc for each $x \in X$. Assume *(iii)* and let $a, b \in px$ with $a \in pb$ and $f(a) = f(b)$. If $c \in ab$, then $a \leqslant_p c \leqslant_p b$, whence $f(a) \leqslant_q f(c) \leqslant_q f(b)$. Since the quasi-order $\leqslant_q$ on the arc $f(p)f(x)$ is a partial order, hence we conclude that $f(a) = f(c) = f(b)$; thus point-inverses of $f \mid px$ are continua, which is equivalent to monotoneity of $f \mid px$ (see [26], (2.2), p. 138).

Second, let X be metric and Y be arcwise connected. By Theorem 4.4.17 of [9], p. 356 the continuum Y is metrizable, and hence $f(p)f(x)$ is a metric arc for each point x of X. Since $f \mid px$ is monotone relative to p (by *(i)* and Proposition 1), we can apply Theorem 1 of [16], § 48, IV, p. 200, which says that the mapping $f \mid px$ can be factored as the composition of two mappings, $f \mid px = h \circ g$, with $g : px \to [0, 1]$ being the canonical Kuratowski monotone mapping ([16], p. 199) and $h : [0, 1] \to f(p)f(x)$. We show that h is monotone. By the equivalence *(i)* $\Leftrightarrow$ *(iii)* (Proposition 2) it is enough to show h is monotone relative to the point 0. Really, let Q be a subcontinuum of the arc $f(p)f(x)$ with $f(p) \in Q$. Now $h^{-1}(Q) = g((f \mid px)^{-1}(Q))$ is connected. Thus $f \mid px$ is monotone as the composition of two monotone mappings, and *(iv)* follows.

As a consequence of Proposition 3 we have the following corollary (cf. [18], Corollary 2.10 *(iii)*, p. 722).

COROLLARY 1. *If a continuum X is an arc with an end point p, and if a mapping f on X is monotone relative to p, then the image $f(X)$ is an arc, and f is monotone.*

In fact, $f(X)$ is hereditarily unicoherent at $f(p)$ by Theorem 2.5 of [18], p. 721. The conclusion follows from the equivalence $(i) \Leftrightarrow (iv)$.

Observe that hereditary unicoherence of the continuum Y at $f(p)$ is a necessary assumption for the implication from (iv) to (i) in Proposition 2. It can seen by the mapping of the unit interval $X = [0, 1]$ with $p = \frac{1}{2}$ onto the unit circle $\{z : |z| = 1\}$ defined by $f(x) = \exp 2\pi ix$ for $x \in X$. Then (iv) holds and (i) does not.

Note further that arcwise connectedness of either X or Y is essential to prove the implication from (iii) to (iv) in Proposition 3. In fact, let X be the $\sin 1/x$-curve mentioned above, and let a mapping f identify the end points $(0, 1)$ and $(0, -1)$ of the limit segemnt. Then the (irreducible) continua X and $f(X)$ are hereditarily unicoherent at $p = (1, \sin 1)$ and $f(p)$ respectively, neither X nor $f(X)$ is arcwise connected, f is $<_p$-preserving but not monotone.

However, the authors do not know if the condition of metrizability of X can be cancelled from the hypotheses of the second part of Proposition 3. In other words we have the following

Problem 1. Let continua X and Y be hereditarily unicoherent at points p and q, respectively, and let a mapping $f : X \to Y$ of X onto Y with $f(p) = q$ be monotone relative to the point p. Does arcwise connectedness of Y imply that $f \mid px$ is monotone for each point x in X?

We close this section recalling that some concept discussed therein were investigated in [13] and [17] from some other point of view.

§ 3. Inverse limits

We use the following notation. $\{X^\lambda, f^{\lambda\mu}, \Lambda\}$ denotes an inverse system of compact spaces X^λ with continuous bonding mappings $f^{\lambda\mu} : X^\mu \to X^\lambda$ for any $\lambda < \mu$, where $\lambda, \mu \in \Lambda$ and Λ is a set directed by the relation $<$. We assume that $f^{\lambda\lambda}$ is the identity, and we denote by $X = \varprojlim \{X^\lambda, f^{\lambda\mu}, \Lambda\}$ the inverse limit space. Further, $\pi^\lambda : X \to X^\lambda$ denotes the projection from the inverse limit space into λ-th factor space. Given a point $p \in X$, we put $p^\lambda = \pi^\lambda(p) \in X^\lambda$; and we write $p = \{p^\lambda\}$. Obviously we have $f^{\lambda\mu}(p^\mu) = p^\lambda$ for any $\lambda, \mu \in \Lambda$ with $\lambda < \mu$. A point $p \in X$, i. e., a system of points $p^\lambda \in X^\lambda$ for $\lambda \in \Lambda$ satisfying the above condition is called a thread.

The following proposition is known (see [6], Corollary 1).

PROPOSITION 4. *Let* $X = \varprojlim \{X^\lambda, f^{\lambda\mu}, \Lambda\}$, *where* X^λ *are continua. If there exists a thread* $p = \{p^\lambda\} \in X$ *such that* X^λ *is hereditarily* *unicoherent at* p^λ *for each* $\lambda \in \Lambda$, *then* X *is a continuum which is hereditarily unicoherent at the point* p.

PROPOSITION 5. *Let* X *be the inverse limit of an inverse system* $\{X^\lambda, f^{\lambda\mu}, \Lambda\}$, *where* X^λ *is a continuum irreducible between some two points* a^λ *and* b^λ, *with* $f^{\lambda\mu}(a^\mu) = a^\lambda$ *and* $f^{\lambda\mu}(b^\mu) = b^\lambda$ *for* $\lambda, \mu \in \Lambda$ *and* $\lambda \leqslant \mu$. *Then* X *is a continuum which is irreducible between the threads* $a = \{a^\lambda\}$ *and* $b = \{b^\lambda\}$.

In fact, X is a continuum as the inverse limit of an inverse system of continua ([9], 6.1.18, p. 436). Let $Y \subset X$ be a continuum containing a and b. Consider the projections $\pi^\lambda | Y : Y \to X^\lambda$, and observe that the image $\pi^\lambda(Y)$ contains a^λ and b^λ, whence $\pi^\lambda(Y) = a^\lambda b^\lambda = X^\lambda$. So Y contains all the threads of the inverse limit X and we have $Y = X$.

Recall that a continuum X is said to be smooth at a point $p \in X$ (in the sense of Gordh, [10], p. 52) if X is hereditarily unicoherent at p and for each convergent net $\{a_n; n \in D\}$ (where D is a directed set) of points of X the condition $\lim a_n = a$ implies that the net $\{pa_n; n \in D\}$ of subcontinua of X is convergent to the limit continuum pa. The point p is then called an initial point of X. The continuum X is said to be smooth if there is an initial point in X.

THEOREM 1. *Let* X *be the inverse limit of an inverse system* $\{X^\lambda, f^{\lambda\mu}, \Lambda\}$, *where* X^λ *are continua. If there exists a thread* $p = \{p^\lambda\}$ *such that the bonding mapping* $f^{\lambda\mu}: X^\mu \to X^\lambda$ *is monotone relative to* p^μ *for each* $\lambda, \mu \in \Lambda$ *with* $\lambda \leqslant \mu$, *and if* X^λ *is smooth at* p^λ *for each* $\lambda \in \Lambda$, *then* X *is a continuum which is smooth at the point* p.

Proof. By Proposition 4 the inverse limit X is a continuum which is hereditarily unicoherent at p. By Theorem 3.1 of [11], p. 65 for each $\lambda \in \Lambda$ there exists a closed weak cutpoint order $\leqslant_\lambda$ with respect to the point p^λ. Define a relation $\leqslant$ on X by $x \leqslant y$ if and only if $x^\lambda \leqslant_\lambda y^\lambda$ for each $\lambda \in \Lambda$. Note that $\leqslant$ is transitive and reflexive, i. e., that it is a quasi-order. We claim that $\leqslant$ is closed. To see this, consider two convergent nets $\{x_n; n \in D\}$ and $\{y_n; n \in D\}$ of points of X having x and y of X as their limits respectively. Assume that $x_n \leqslant y_n$ for each $n \in D$. Thus $x_n^\lambda = \pi^\lambda(x_n) \leqslant_\lambda y_n^\lambda = \pi^\lambda(y_n)$ for each $\lambda \in \Lambda$ by the definition of the quasi-order $\leqslant$ on X. Since the net $\{x_n; n \in D\}$ of points of the inverse limit space X converges to the limit point x if and only if the nets $\{x_n; n \in D\}$ converge to x^λ for each $\lambda \in \Lambda$ (see [9], 2.3.34, p. 119), and since each quasi-order $\leqslant_\lambda$ is closed, we have $x^\lambda \leqslant_\lambda y^\lambda$ for each $\lambda \in \Lambda$, and thus the claim is proved.

To complete the proof we only ought to show the quasi-order defined above is just the weak cutpoint order with respect to p, i. e., that $x \leqslant y$ holds if and only if $px \subset py$. To this end let us take for $\lambda \leqslant \mu$ the partial mapping $g^{\lambda\mu} = f^{\lambda\mu} | p^\mu y^\mu$ from the unique irreducible continuum $p^\mu y^\mu$ into the continuum X^λ.

It follows from Theorem 2.5 (*ii*) of [18], p. 721 that $g^{\lambda\mu}(p^{\mu}y^{\mu}) = p^{\lambda}y^{\lambda}$, so we can consider the mapping $g^{\lambda\mu}: p^{\mu}y^{\mu} \to p^{\lambda}y^{\lambda}$ as a surjection. Further, it is monotone relative to p^{μ} by Proposition 3. Obviously we have $g^{\lambda\mu}(p^{\mu}) = p^{\lambda}$ and $g^{\lambda\mu}(y^{\mu}) = y^{\lambda}$. Since $\{\pi^{\lambda}(X), f^{\lambda\mu}| \pi^{\mu}(X), \Lambda\}$ is an inverse system ([3], 2.8, p. 235), it can be easily verified that $\{p^{\lambda}y^{\lambda}, g^{\lambda\mu}, \Lambda\}$ is an inverse system. Let L denote its inverse limit. Obviously $p, y \in L$. Further, L is a continuum which is irreducible from p to y by Proposition 5. Thus $L = py$, since by hereditary unicoherence of X at p there is only one continuum irreducible from p to y ([10], Theorem 1.3, p. 52).

Furthermore, since $\{p^{\lambda}y^{\lambda}, g^{\lambda\mu}, \Lambda\}$ is an inverse system, we conclude that the condition $x \leqslant y$, i. e., $x^{\lambda} \leqslant_{\lambda} y^{\lambda}$ for each $\lambda \in \Lambda$ (which means that $p^{\lambda}x^{\lambda} \subset p^{\lambda}y^{\lambda}$ for each $\lambda \in \Lambda$), is equivalent to $px \subset py$. So the proof is finished.

As a consequence of Theorem 1 we have (cf. [6], Theorem 1) the following

COROLLARY 2. *Let X be the inverse limit of an inverse system $\{X^{\lambda}, f^{\lambda\mu}, \Lambda\}$, where X^{λ} are continua and $f^{\lambda\mu}$ are monotone mappings. If there exists a thread $p = \{p^{\lambda}\}$ such that X^{λ} is smooth at p^{λ} for each $\lambda \in \Lambda$, then X is a continuum which is smooth at the point p.*

By an arboroid (see e. g. [17], p. 166) we mean a continuum which is arcwise connected and hereditarily unicoherent (i. e. any two points of the continuum can be joined by a (generalized) arc and the intersection of any two subcontinua is a continuum). A metrizable arboroid is called a dendroid.

THEOREM 2. *Let X denote an inverse limit of an inverse system $\{X^{\lambda}, f^{\lambda\mu}, \Lambda\}$, where 1° X^{λ} is an arboroid for each $\lambda \in \Lambda$, and 2° there exists a thread $p = \{p^{\lambda}\}$ such that the mapping $f^{\lambda\mu}$ is monotone relative to p^{μ} for each $\lambda, \mu \in \Lambda$ with $\lambda \leqslant \mu$. Then X is an arboroid.*

Proof. By Proposition 4 the continuum X is hereditarily unicoherent. We prove that for each point $x \in X$ the irreducible continuum px is an arc. Note that the projections $\pi^{\lambda}: X \to X^{\lambda}$ are monotone relative to p by Theorem 1 of [7], p. 4, so $\pi^{\lambda}(px) = \pi^{\lambda}(p)\,\pi^{\lambda}(x) = p^{\lambda}x^{\lambda}$ by (*ii*) of Proposition 2. Since X^{λ} is an arboroid, $p^{\lambda}x^{\lambda}$ is an arc. Thus we have $px = \underleftarrow{\lim}\{p^{\lambda}x^{\lambda}, f^{\lambda\mu}\,|\,p^{\mu}x^{\mu}, \Lambda\}$ by [3], 2.8, p. 235. Since $f^{\lambda\mu}\,|\,p^{\mu}x^{\mu}$ are monotone relative to p^{μ} by Proposition 1, these mappings are monotone by Corollary 1, thus we conclude that px is an arc as the inverse limit of an inverse system of arcs with monotone bonding mappings (see [19]; cf. [12], p. 416).

Since the limit of an inverse sequence of metrizable spaces is metrizable ([9], Corollary 4.2.5, p. 325) we have

COROLLARY 3. *Let X denote an inverse limit of an inverse sequence $\{X^{i}, f^{i}\}_{i=1}^{\infty}$, where 1° X^{i} is a dendroid, and 2° there exists a*

It follows from Theorem 2.5 (*ii*) of [18], p. 721 that $g^{\lambda\mu}(p^{\mu}y^{\mu}) = p^{\lambda}y^{\lambda}$, so we can consider the mapping $g^{\lambda\mu}: p^{\mu}y^{\mu} \to p^{\lambda}y^{\lambda}$ as a surjection. Further, it is monotone relative to p^{μ} by Proposition 3. Obviously we have $g^{\lambda\mu}(p^{\mu}) = p^{\lambda}$ and $g^{\lambda\mu}(y^{\mu}) = y^{\lambda}$. Since $\{\pi^{\lambda}(X), f^{\lambda\mu}|\pi^{\mu}(X), \Lambda\}$ is an inverse system ([3], 2.8, p. 235), it can be easily verified that $\{p^{\lambda}y^{\lambda}, g^{\lambda\mu}, \Lambda\}$ is an inverse system. Let L denote its inverse limit. Obviously $p, y \in L$. Further, L is a continuum which is irreducible from p to y by Proposition 5. Thus $L = py$, since by hereditary unicoherence of X at p there is only one continuum irreducible from p to y ([10], Theorem 1.3, p. 52).

Furthermore, since $\{p^{\lambda}y^{\lambda}, g^{\lambda\mu}, \Lambda\}$ is an inverse system, we conclude that the condition $x \leqslant y$, i. e., $x^{\lambda} \leqslant_{\lambda} y^{\lambda}$ for each $\lambda \in \Lambda$ (which means that $p^{\lambda}x^{\lambda} \subset p^{\lambda}y^{\lambda}$ for each $\lambda \in \Lambda$), is equivalent to $px \subset py$. So the proof is finished.

As a consequence of Theorem 1 we have (cf. [6], Theorem 1) the following

COROLLARY 2. *Let* X *be the inverse limit of an inverse system* $\{X^{\lambda}, f^{\lambda\mu}, \Lambda\}$, *where* X^{λ} *are continua and* $f^{\lambda\mu}$ *are monotone mappings. If there exists a thread* $p = \{p^{\lambda}\}$ *such that* X^{λ} *is smooth at* p^{λ} *for each* $\lambda \in \Lambda$, *then* X *is a continuum which is smooth at the point* p.

By an arboroid (see e. g. [17], p. 166) we mean a continuum which is arcwise connected and hereditarily unicoherent (i. e. any two points of the continuum can be joined by a (generalized) arc and the intersection of any two subcontinua is a continuum). A metrizable arboroid is called a dendroid.

THEOREM 2. *Let* X *denote an inverse limit of an inverse system* $\{X^{\lambda}, f^{\lambda\mu}, \Lambda\}$, *where* 1° X^{λ} *is an arboroid for each* $\lambda \in \Lambda$, *and* 2° *there exists a thread* $p = \{p^{\lambda}\}$ *such that the mapping* $f^{\lambda\mu}$ *is monotone relative to* p^{μ} *for each* $\lambda, \mu \in \Lambda$ *with* $\lambda \leqslant \mu$. *Then* X *is an arboroid.*

Proof. By Proposition 4 the continuum X is hereditarily unicoherent. We prove that for each point $x \in X$ the irreducible continuum px is an arc. Note that the projections $\pi^{\lambda}: X \to X^{\lambda}$ are monotone relative to p by Theorem 1 of [7], p. 4, so $\pi^{\lambda}(px) = \pi^{\lambda}(p)\,\pi^{\lambda}(x) = p^{\lambda}x^{\lambda}$ by (*ii*) of Proposition 2. Since X^{λ} is an arboroid, $p^{\lambda}x^{\lambda}$ is an arc. Thus we have $px = \varprojlim\{p^{\lambda}x^{\lambda}, f^{\lambda\mu}|p^{\mu}x^{\mu}, \Lambda\}$ by [3], 2.8, p. 235. Since $f^{\lambda\mu}|p^{\mu}x^{\mu}$ are monotone relative to p^{μ} by Proposition 1, these mappings are monotone by Corollary 1, thus we conclude that px is an arc as the inverse limit of an inverse system of arcs with monotone bonding mappings (see [19]; cf. [12], p. 416).

Since the limit of an inverse sequence of metrizable spaces is metrizable ([9], Corollary 4.2.5, p. 325) we have

COROLLARY 3. *Let* X *denote an inverse limit of an inverse sequence* $\{X^{i}, f^{i}\}_{i=1}^{\infty}$, *where* 1° X^{i} *is a dendroid, and* 2° *there exists a* *thread* $p = \{p^i\}$ *such that the mapping* f^i *is monotone relative to* p^{i+1}, *for each* $i = 1, 2, \ldots$. *Then* X *is a dendroid.*

Corollary 3 generalizes a result saying that the inverse limit of an inverse sequence of dendroids with monotone bonding mappings is a dendroid (see [2], Lemma 1, p. 192 and [6], Theorem 2).

Since an arboroid is smooth if and only if it is a generalized tree ([15]), Theorem, p. 680), Theorems 1 and 2 imply

COROLLARY 4. *Let* X *be the inverse limit of an inverse system* $\{X^\lambda, f^{\lambda\mu}, \Lambda\}$, *where* 1° X *is a generalized tree with a point* p^λ *as a zero*; 2° *the points* p^λ *from a thread* $p = \{p^\lambda\}$; *and* 3° *the mapping* $f^{\lambda\mu}$ *is monotone relative to* p^μ *for each* $\lambda, \mu \in \Lambda$ *with* $\lambda \leqslant \mu$. *Then* X *is a generalized tree with the point* p *as a zero.*

A point p of an arboroid X is said to be a ramification point if there are three arcs in x emanating from p and disjoint out of p. An arboroid having at most one ramification point is called a fan. Then the ramification point is called the top of the fan.

Using Corollary 1 one can prove — by exactly the same methods as that in the proof of Theorem 3 of [6] — the following result.

THEOREM 3. *Let* X *denote an inverse limit of an inverse system* $\{X^\lambda, f^{\lambda\mu}, \Lambda\}$, *where* 1° X^λ *is a fan with the top* p^λ *for each* $\lambda \in \Lambda$; 2° *the points* p^λ *form a thread* $p = \{p^\lambda\}$; *and* 3° *the mapping* $f^{\lambda\mu}$ *is monotone relative to* p^μ *for each* $\lambda, \mu \in \Lambda$ *with* $\lambda \leqslant \mu$. *Then* X *is a fan with the top* p.

Combining Theorem 3 and Corollary 4 we get

COROLLARY 5. *Let* X *be the inverse limit of an inverse system* $\{X^\lambda, f^{\lambda\mu}, \Lambda\}$, *where* 1° X^λ *is a fan with the top* p^λ *at which it is smooth*; 2° *the points* p^λ *form a thread* $p = \{p^\lambda\}$; *and* 3° *the mapping* $f^{\lambda\mu}$ *is monotone relative to* p^μ *for each* $\lambda, \mu \in \Lambda$ *with* $\lambda \leqslant \mu$. *Then* X *is a fan which is smooth at its top* p.

§ 4. Examples and remarks

Further considerations concern essentiality of some assumptions in the theorems and in the corollaries. Moreover, some other properties are discussed in connection with the inverse limit operation. We start with

Example 1. Let X be the simplest Knaster indecomposable continuum with one end point (sec [16], § 48, V, Example 1, p. 204 and Fig. 4, p. 205). It is well known that X is the inverse limit of an inverse sequence $\{X^i, f^i\}_{i=1}^{\infty}$ of closed unit intervals $X^i = [0, 1]$ with the same mappings $f^i : X^{i+1} \to X^i$ defined by $f^i(t) = 1 - |2t - 1|$ for $t \in [0, 1]$ and $i = 1, 2, \ldots$. It is easy to note that each X^i is smooth at any point, in particular at $t = \frac{1}{2}$, each mapping f^i is monotone relative to $t = \frac{1}{2}$, but $f^i\left(\frac{1}{2}\right) \neq \frac{1}{2}$ for each $i = 1, 2, \ldots$ (thus the points that the mappings f^i are monotone relative to them do not form a thread) and the continuum X is smooth at no point.

It follows from the above example that smoothness of continua is not preserved under the inverse limit operation with bonding mappings being monotone relative to points if these points do not form a thread. Thus the corresponding assumption in Theorem 1 is essential.

The same example shows essentiality of this assumption in Theorem 2, and in Corollary 4.

However, the reader can conjecture that the conclusion of Theorem 1 holds under a weaker assumption, namely that there are two (not necessarily the same) threads: one composed of points of smoothness of the continua X^λ, and the other composed of points that the mappings $f^{\lambda\mu}$ are monotone relative to them. The next example shows that it is not the case.

Example 2. Let the Euclidean plane equipped with a rectangular coordinate system (x, y) be given. Let A be the straight line segment of y-axis with the end points (0,1) and $(0, -1)$, and let A_n and B_n denote the straight line segments joining the points (0, 1) with $\left(\frac{1}{n}, -1\right)$ and $(0, -1)$ with $\left(-\frac{1}{n}, 1\right)$ respectively. Then $H = A \cup \bigcup\{A : n = 1, 2, \ldots\}$ is a harmonic fan. Define $X^i = H \cup \bigcup\{B_n : n = 1, 2, \ldots i\}$ and note that X^i is a dendroid which is smooth at the point (0, 1), and that we have $X^1 \subset X^2 \subset \ldots \subset X^i \subset X^{i+1} \subset \ldots$. Further, defi-ne $f^i : X^{i+1} \to X^i$ putting

$$f^i((x, y)) = \begin{cases} (0, y), & \text{if } (x, y) \in B_{i+1}, \\ (x, y) & \text{otherwise.} \end{cases}$$

Thus $f^i \mid B_{i+1}$ is the projection (parallel to the x-axis) of B_{i+1} onto A and $f^i \mid X^{i+1} \setminus B_{i+1}$ is the identity. Observe that $f^i((0, 1)) = (0, 1)$ for every $i = 1, 2, 3, \ldots$, whence we conclude that the points $p^i = (0, 1)$ of smoothness of X^i form a thread $p = \{p^i\}_{i=1}^\infty$. It can be easily verified (e. g. using (*iv*) of Proposition 1) that the mappings f^i are monotone relative to the points $q^{i+1} = (0, -1) \in X^{i+1}$ for every $i = 1, 2, \ldots$; since $f^i((0, -1)) = (0, -1)$, the points q^i form a thread $q = \{q^i\}_{i=1}^\infty$.

By a routine argumentation (using e. g. Theorem I of [1], p. 348) one can see that $\{X^i, f^i\}_{i=1}^\infty$ is an inverse system whose limit X is homeomorphic to $H \cup \bigcup\{B_n : n = 1, 2, \ldots\} = \cup\{X^i : i = 1, 2, \ldots\} = \operatorname{Lim} X^i$ (here Lim denotes the topological limit). Omitting the homeomorphism for shortness we see that $X = A \cup \bigcup \{A_n \cup B_n : n = 1, 2, \ldots\}$ is a dendroid with $p = (0, 1)$ and $q = (0, -1) \neq p$ which is not smooth.

A space X is said to be contractible if there exists a homotopy $H : X \times [0, 1] \to X$ such that $H(x, 0) = x$ for all $x \in X$ and $H(X \times \{1\})$ is a singleton. It is known that every contractible curve (i. e. a one-dimensional metric continuum) is a dendroid ([4], Proposition 1, p. 73).

Example 2 shows that contractibility of dendroids is not preserved under the inverse limit operation if the bonding mappings are monotone relative to some points. In fact, the dendroids X^i are obviously contractible for $i = 1, 2, \ldots$ while $X = \lim \{X^i, f^i\}$ is not, because it is of type N between the points p and q (see [22], Theorem 2.1, p. 838).

The problem of finding some (necessary and/or sufficient) conditions under which the inverse limit of an inverse system of contractible continua is contractible remains open. In particular we have

Problem 2. Let X be the inverse limit of an inverse sequence $\{X^i, f^i\}_{i=1}^{\infty}$ such that 1° each X^i is a contractible dendroid (fan), and 2° each f^i is monotone. Is then the dendroid (fan) X contractible?

A continuum X is said to be selectible ([5], p. 109) if the hyperspace $C(X)$ of its subcontinua (equipped with the Vietoris topology) admits a continuous selection. It is known ([21], Lemma 3, p. 370; cf. [5], Proposition 1, p. 110) that each metrizable selectible continuum is a dendroid.

Similarly to contractibility of dendroids discussed above, Example 2 shows that selectibility is not transformed from factors to the inverse limit space for inverse sequences of dendroids with bonding mappings being — as previously — monotone relative to some points which form a thread. Indeed, since each X^i is smooth, it is selectible (see ([25], Theorem 2, p. 1043), but X is not ([21], Theorem 2, p. 372). A natural question arises of finding some conditions under which the inverse limit of an inverse system of selectible dendroids is a selectible dendroid. In particular we have

Problem 3. Assume X is the inverse limit of an inverse sequence $\{X^i, f^i\}_{i=1}^{\infty}$ of selectible dendroids (fans) X^i with monotone bonding mappings f^i. Is then the dendroid (fan) X selectible?

S. B. Nadler, Jr. has shown in [20], Theorem 4 (part 3), p. 229 that the property of being a dendrite (i. e. a locally connected dendroid) is an invariant of the inverse limit operation for inverse sequences with monotone bonding mappings. In contrast to the corresponding results for dendroids (see [2], Lemma 1, p. 192 and Corollary 2 above), monotoneity of the bonding mappings cannot be relaxed to one relative to some points even if these points form a thread. This can be seen from

Example 3. In the Euclidean plane equipped with a rectangular coordinate system we take the Cantor fan X specified as the cone with the top $\left(\frac{1}{2}, \frac{1}{2}\right)$ over the Cantor ternary set C lying in the interval $[0, 1]$ of the x-axis. Let E^i be the set composed of 2^i points of C, namely of $(0, 0)$, $(1, 0)$, and of end points of the contiguous intervals with lengths $1/3$, $1/3^2, \ldots, 1/3^i$.

Let X^i denote the cone over E^i with the top $\left(\frac{1}{2}, \frac{1}{2}\right)$. Thus we have

$$X^1 \subset X^2 \subset \ldots \subset X^i \subset X^{i+1} \subset \ldots \subset X,$$

and we see that each X^i is a finite dendrite (even a finite fan), so it is a regular curve, while X is not rational even (in the sense of the theory of order, see [16], § 51, I, p. 275).

Define $f^i : X^{i+1} \to X^i$ as a natural retraction, i. e., we take $f^i|X^i$ as the identity and we let $f^i \mid X^{i+1} \setminus X^i$ be the projection parallel to the x-axis of each straight line segment joining $\left(\frac{1}{2}, \frac{1}{2}\right)$ with a point of $E^{i+1} \setminus E^i$ onto the nearest straight line segment from $\left(\frac{1}{2}, \frac{1}{2}\right)$ to a corresponding point of $E^i \setminus E^{i-1}$. Thus by (*iv*) of Proposition 1 each f^i is monotone relative to $p^{i+1} = \left(\frac{1}{2}, \frac{1}{2}\right)$, the points p^i form a thread and the inverse limit is homeomorphic to X (again by Theorem I of [1], p. 348).

It follows from Corollary 3 and Theorem 1 that the inverse limit of an inverse sequence of dendrites with bonding mappings being monotone relative to points which form a thread is a smooth dendroid. In a recent paper [27] the second of the authors proved an inverse result: every smooth dendroid is the inverse limit of an inverse sequence of finite dendrites with bonding mappings being monotone relative to points which form a thread.

Note that the thread $\{p^i\}$ in Example 3 corresponds to the top $\left(\frac{1}{2}, \frac{1}{2}\right)$ of the Cantor fan X at which X is locally connected. In connection to this fact we have

Problem 4. Let X be the inverse limit of inverse sequence $\{X^i, f^i\}_{i=1}^{\infty}$ such that 1° X^i is a metric continuum; 2° X^i is locally connected at a point p^i; 3° the points p^i form a thread $p = \{p^i\}$; and 4° the bonding mappings f^i are monotone relative to p^{i+1}. Does it follow that the continuum X is locally connected at the point p?

REFERENCES:

[1] *R. D. Anderson* and *G. Choquet*, A plane continuum no two of whose nondegenerate subcontinua are homeomorphic: an application of inverse limits, Proc. Amer. Math. Soc. **10** (1959), 347—353.

[2] *D. P. Bellamy*, An interesting plane dendroid, Fund. Math. **110** (1980), 191—208

[3] *C. E. Capel*, Inverse limit spaces, Duke Math. J. **21** (1954), 233—245.

[4] *J. J. Charatonik*, Problems and remarks on contractibility of curves, General Topology and its Relations to Modern Analysis and Algebra IV, Proceedings of the Fourth Prague Topological Symposium, 1976, Part B, Contributed Papers, Society of Czechoslovak Mathematicians and Physicists, Prague 1977, pp. 72—76.

[5] ————, Contractibility and continuous selections, Fund. Math. **108** (1980), 109—118.

[6] *J. J. Charatonik* and *W. J. Charatonik*, Inverse limits and smoothness of continua, Acta Math. Hungar. **43** (1984), 7—12.

[7] ————————, On projections and limit mappings of inverse systems of compact spaces, Topology Appl. **16** (1983), 1—9.

[8] *J. J. Charatonik* and *C. Eberhart*, On smooth dendrois, Fund. Math. **67** (1970), 297—322.

[9] *R. Engelking*, General topology, PWN, Warsaw 1977.

[10] *G. R. Gordh, Jr.*, On decompositions of smooth continua, Fund. Math. **75** (1972), 51—60.

[11] ————, Concerning closed quasi-orders on hereditarily unicoherent continua, Fund. Math. **78** (1973), 61—73.

[12] *G. R. Gordh, Jr.* and *S. Mardešić*, Characterizing local connectedness in inverse limits, Pacific J. Math. **58** (1975), 411—417.

[13] *G. R. Gordh, Jr.* and *L. Lum*, Radially convex mappings and smoothness of continua, Houston J. Math. **4** (1978), 335—342.

[14] *J. G. Hocking* and *G. S. Young*, Topology, Reading-London 1961.

[15] *R. J. Koch* and *I. S. Krule*, Weak cut-point ordering on hereditarily unicoherent continua, Proc. Amer. Math. Soc. **11** (1960), 679—681.

[16] *K. Kuratowski*, Topology II, Academic Press and PWN, 1968.

[17] *L. Lum*, Weakly smooth continua, Trans. Amer. Math. Soc. **214** (1975), 153—167.

[18] *T. Maćkowiak*, Confluent mappings and smoothness of dendroids, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys. **21** (1973), 719—725.

[19] *S. Mardešić*, Locally connected, ordered and chainable continua, Rad Jugoslav. Akad. Znan. Umjetn. **319** (1960), 147—166.

[20] *S. B. Nadler, Jr.*, Multicoherence techniques applied to inverse limits, Trans. Amer. Math. Soc. **157** (1971), 227—234.

[21] *S. B. Nadler, Jr.* and *L. E. Ward, Jr.*, Concerning continuous selections, Proc. Amer. Math. Soc. **25** (1970), 369—374.

[22] *L. G. Oversteegen*, Non-contractibility of continua, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys. **26** (1978), 837—840.

[23] *A. J. Ward*, Notes on general topology II: A generalization of arc-connectedess, Proc. Cambridge Philos. Soc. **61** (1965), 879—880.

[24] *L. E. Ward, Jr.*, Mobs, trees nad fixed points. Proc. Amer. Math. Soc. **8** (1957), 798—804.

[25] ——————————, Rigid selections on smooth dendroids, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys. **19** (1971), 1041—1043.
[26] *G. T. Whyburn,* Analytic topology, AMS Colloq. Publications Vol. 28, 1942.
[27] *W. J. Charatonik,* Smooth dendroids as inverse limits of dendrites, Fund. Math. **124** (1984), 163—168.

(Received May 28, 1982)
(Revised November 1, 1983)

Institute of Mathematics
University of Wrocław
pl. Grunwaldzki 2/4
50-384 *Wrocław, Poland*

MONOTONOST S OBZIROM NA TOČKU I INVERZNI LIMESI KONTINUUMA

J. J. Charatonik i *W. J. Charatonik*, Wrocław, Poljska

Sadržaj

U članku je dokazano da je inverzni limes inverznog sistema glatkih kontinuuma glatki kontinuum ako postoji nit koja se sastoji od inicijalnih točaka i ako su vezna preslikavanja monotona s obzirom na te točke. Također je dokazano da se neka svojstva prenose sa članova inverznog sistema na inverzni limes ako inverzni sistem ima gore navedena svojstva.