Mappings between inverse limits of continua with multivalued bonding functions

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**Abstract**

We investigate the limit mappings between inverse limits of continua with upper semi-continuous bonding functions. Results are obtained when the coordinate mappings are surjective, one-to-one or homeomorphisms. We construct examples showing the hypothesis of the theorems are essential. Further, we construct an example showing that, unlike for the inverse limits with single valued maps, properties of being monotone, confluent or weakly confluent mappings between factor spaces are not preserved in the inverse limit map.

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1. Introduction

Consider two inverse systems \(\{X_\lambda, f^\mu_\lambda, \Lambda\}\) and \(\{Y_\sigma, g^\tau_\sigma, \Sigma\}\) and let \(\{\phi, h_\sigma\}\) be a mapping between those inverse systems. It was shown in [3] and [6], that if all coordinate mappings \(h_\sigma : X_{\phi(\sigma)} \rightarrow Y_\sigma\) are monotone, then the limit mapping \(\lim \left\{\sigma, h_\sigma\right\} : \lim \left\{X_\lambda, f^\mu_\lambda, \Lambda\right\} \rightarrow \lim \left\{Y_\sigma, g^\tau_\sigma, \Sigma\right\}\) is also monotone. Similarly, it was shown in [1] that if the coordinate mappings \(h_\sigma\) are confluent or weakly confluent, then the limit mapping is confluent or weakly confluent, respectively.

In their paper [5, Theorem 5.3, p. 126], Ingram and Mahavier showed that if \(X\) and \(Y\) are inverse limits of inverse sequences with upper semi-continuous set valued bonding maps and each \(h_i : X_i \rightarrow Y_i\) is a homeomorphism then \(\lim h_i\) is a homeomorphism. In this note we show, under suitable assumptions, that if each of the \(h_\sigma\)’s is surjective, one-to-one, or homeomorphisms then \(\lim h_\sigma\) is also surjective, one-to-one, or a homeomorphism. Further we give an example to show that the \(h_\sigma\)’s can all be monotone, thus confluent and weakly confluent, without \(\lim h_\sigma\) having to be even weakly confluent.

2. Preliminaries

By a map or mapping we mean a continuous function. We will consider inverse systems \(\{X_\lambda, f^\mu_\lambda, \Lambda\}\), where \(\Lambda\) is a directed set, for each \(\lambda \in \Lambda, X_\lambda\) is a topological space, and for each two indices \(\lambda, \mu \in \Lambda\) satisfying \(\lambda \leq \mu\) the multivalued function \(f^\mu_\lambda : X_\mu \rightarrow X_\lambda\) is upper semi-continuous. Moreover we assume that \(f^\mu_\lambda\) is the identity function on \(X_\lambda\) and that, for every triple \(\lambda, \mu, \nu \in \Lambda\) satisfying \(\lambda \leq \mu \leq \nu\), we have \(f^\nu_\lambda \subseteq f^\mu_\lambda \circ f^\nu_\mu\). In this setting we may define the inverse limit space...
\[ \text{lim}(X, F^\mu F^\nu, \Lambda) = \{ x : \Lambda \to \prod_{x_\lambda \in x} X_\lambda : x(\lambda) \in X_\lambda \text{ for } \lambda \in \Lambda \text{, and } x(\lambda) \in F^\mu (x(\mu)) \text{ for each } \lambda < \mu \}. \]

The members of the inverse limits are called threads and we will use the index notation \( x_\lambda \) rather than \( x(\lambda) \). This approach is more general then the original in [5], not only by considering arbitrary directed sets as the set of indices, but mainly because the commutativity condition \( F^\nu F^\mu = F^{\mu \nu} \) was replaced by the inclusion \( F^\nu F^\mu \subseteq F^{\mu \nu} \).

Suppose two inverse systems \( S = (X_\lambda, F^\mu_\lambda, \Lambda) \) and \( T = (Y_\sigma, G^\sigma_\sigma, \Sigma) \) are given. By a mapping of \( S \) to \( T \) we mean a family \( \{ \sigma, \eta_\sigma \} \) consisting of nondecreasing function \( \phi : \Sigma \to \Lambda \) such that the set \( \phi(\Sigma) \) is cofinal in \( \Lambda \) and of continuous (single valued) mappings \( h_\sigma : X_{\phi(\sigma)} \to Y_\sigma \) such that \( h_\rho \circ F^\phi_\phi(\sigma) \subseteq G^\sigma_\rho h_\sigma \). Any mapping of \( S \) to \( T \) induces a limit mapping \( h = \text{lim}(\phi, \eta) : \text{lim}S \to \text{lim}T \) defined by \( h(x) = h_\sigma(x_{\phi(\sigma)}) \). This definition generalizes the definition of a limit map in the case of inverse limits with single valued bonding maps (see e.g. [2, p. 101]).

A continuum is a compact and connected space. A map \( f : X \to Y \) between continua \( X \) and \( Y \) is called

- monotone if for every subcontinuum \( C \) of \( Y \) the preimage \( f^{-1}[C] \) is connected;
- confluent if for every subcontinuum \( C \) of \( Y \) and every component \( K \) of \( f^{-1}[C] \) we have \( f(K) = C \); and
- weakly confluent if for every subcontinuum \( C \) of \( Y \) there is a component \( K \) of \( f^{-1}[C] \) such that \( f(K) = C \).

3. Main results

The following theorem generalizes a respective result for single valued bonding functions, see [2, Theorem 3.2.14].

**Theorem 3.1.** Let \( X = \text{lim}(X_\lambda, F^\mu_\lambda, \Lambda) \) and \( Y = \text{lim}(Y_\sigma, G^\sigma_\sigma, \Sigma) \) where each of the spaces \( X_\lambda \) and \( Y_\sigma \) is compact and each of the functions \( F^\mu_\lambda \) and \( G^\sigma_\sigma \) is upper semi-continuous. Further suppose that \( \Sigma \) is a directed set and \( \Lambda \) is linearly ordered, \( \phi : \Sigma \to \Lambda \) is an order preserving function such that \( \phi(\Sigma) \) is cofinal in \( \Lambda \) and that, for each \( \sigma \in \Sigma \), the function \( h_\sigma : X_{\phi(\sigma)} \to Y_\sigma \) is a surjective mapping such that \( h_\sigma \circ F^\phi_\phi(\sigma) \subseteq G^\sigma_\sigma h_\sigma \). Then \( \text{lim}(\phi, \eta) \) is surjective.

**Proof.** Let \( h = \text{lim}(\phi, \eta) \) and take \( y \in Y \). We will show that \( h^{-1}(y) \neq \emptyset \). To this aim define, for finite subsets \( A \subseteq \Sigma \) and \( B \subseteq \Lambda \) with \( \max(\phi(A)) \geq b \) for all \( b \in B \) a set

\[ P_{A,B} = \left\{ x \in \prod_{\lambda \in \Lambda} X_\lambda : h_\sigma(x_{\phi(\sigma)}) = y_\sigma \text{ for all } \sigma \in A \text{ and } x_\lambda \in F^\mu_\lambda(x_{\mu}) \text{ for all } \lambda \leq \mu \text{ where } \lambda, \mu \in \phi(A) \cup B \right\}. \]

One can verify that the set \( P_{A,B} \) is compact. We will show that it is nonempty.

We will proceed by the induction on the number of elements in \( A \). Initially assume \( \phi(A) \cup B = \phi(\Lambda) \). If \( A \) is a one element set \( A = \{ a \} \), then \( P_{A,B} = \{ x \in \prod_{\lambda \in \Lambda} X_\lambda : h_{\phi(a)}(x_{\phi(a)}) = y_a \} \) and it is nonempty, because \( h_a \) is surjective.

Next, suppose for this case, that \( P_{A,B} \) is nonempty for all sets \( A \) having \( n \) elements. Let \( A' = \{ \sigma_1, \ldots, \sigma_{n+1} \} \subseteq \Sigma \) where indexing is chosen so that if \( j > i \) then \( \phi(\sigma_j) \geq \phi(\sigma_i) \). Let \( x \in P_{A' \setminus \{ \sigma_i \}, B} \), then \( h_{\sigma_1}(F^\phi_{\phi(\sigma_1)}(x_{\phi(\sigma_1)})) = G^\sigma_{\sigma_1}(h_{\sigma_1}(x_{\phi(\sigma_1)})) = G^\sigma_{\sigma_1}(y_{\sigma_1}) = y_{\sigma} \), so \( P_{A',B} \neq \emptyset \).

If \( \phi(\Lambda) \cap B = \phi(\Lambda) \cap B \neq \phi(\Lambda) \) take \( x \in P_{\phi(\Lambda) \cap B} \), modify coordinates of \( x \) on \( B \setminus \phi(\Lambda) \) as necessary, using induction on the number of elements in \( B \setminus \phi(\Lambda) \) and the inclusion \( F^\phi_{\phi(\sigma)}(x_{\lambda}) \subseteq F^\phi_{\phi(\sigma)}(x_{\lambda}) \) whenever \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \) to show that \( P_{A,B} \) is nonempty in this case.

Now observe that the sets \( P_{A,B} \) have the finite intersection property so \( \bigcap \{ P_{A,B} : A \text{ and } B \text{ are finite subsets of } \Sigma \text{ and } \Lambda \text{ and } \max(\phi(A)) \geq \min(B) \} \) is nonempty. This intersection is \( h^{-1}(y) \) thus \( h^{-1}(y) \neq \emptyset \).

**Corollary 3.2.** Let \( X = \text{lim}(X_\lambda, F^\mu_\lambda, \Lambda) \) where each of the spaces \( X_\lambda \) is compact and each of the functions \( F^\mu_\lambda \) is upper semi-continuous. Further suppose that \( \Lambda \) is a linearly ordered set. Then \( X \neq \emptyset \).

**Proof.** Let \( Y = \text{lim}(Y_\sigma, G^\sigma_\sigma, \Lambda) \) where for each \( \lambda \in \Lambda \), \( Y_\lambda = \{ y \} \), \( G^\sigma_\sigma \), and \( \phi : \Lambda \to \Lambda \) are identity maps and \( h_\lambda : X_\lambda \to Y_\lambda \) is the constant map. Thus the hypothesis of Theorem 3.1 is satisfied, \( Y = \{ y \} \) so \( X \) is nonempty since it equals the nonempty set \( h^{-1}(y) \).

In our first example we show that the exact commutativity, \( h_1 \circ F_1 = G_1 \circ F_{i+1} \) in Theorem 3.1 is necessary.

**Example 3.3.** Let \( X_i = Y_i \in [0,1] \) for all \( i \in \{ 1,2,\ldots \} \), and define \( h_i(x) = F^1_i(x) = x \) for all \( x \in [0,1] \) and for all \( i \leq j \). Finally, for all \( i \in \{ 1,2,\ldots \} \) and \( x \in [0,1] \), let \( G_i(x) = [0,1] \). Then each of the maps \( h_i \) is surjective and \( h_i \circ F_i(x) \subseteq G_i \circ F_{i+1}(x) \) but \( \text{lim}(X_i, F_i) \) is an arc while \( \text{lim}(Y_i, G_i) \) is the Hilbert cube. So the limit map \( \text{lim} h_i \) between the inverse limit spaces is not surjective.
It is known that without compactness the inverse limit may be empty even for single valued bonding mappings [4], see also [6, Example 1, p. 58].

In the following example we will show that the condition that Λ is linearly ordered in Theorem 3.1 and Corollary 3.2 cannot be relaxed.

Example 3.4. Let Λ = {a, b, c, d, e} such that a ≤ b ≤ c ≤ d ≤ e, a ≤ d, c ≤ b, a ≤ e, and c ≤ e. Then Λ is a directed set. Let X₀ = {p} be a singleton and Xₙ = S¹ for n ∈ Λ \ {e}. Define Fⁿ₁(p) = Fⁿ₂(p) = S¹, Fⁿ₃ = Fⁿ₄ = id and Fⁿ₅ is a rotation by π. If x is a point in lim[Xₙ, F⁻¹ₙ, Λ] then x₀ = x₀ = x and xₖ = xₖ = −x₅, a contradiction. Thus lim[Xₙ, F⁻¹ₙ, Λ] is empty.

Theorem 3.5. Let X = lim[Xₙ, F⁻¹ₙ, Λ] and Y = lim[Yₙ, G⁻¹ₙ, Σ] where each of the functions F⁻¹ₙ and G⁻¹ₙ is upper semi-continuous. Further suppose that Σ and Λ are directed sets, φ : Σ → Λ is order preserving and surjective and that, for each σ ∈ Σ, the function hₜ : Xₚ(φ(σ)) → Yₚ is one-to-one such that, for each σ, τ in Σ satisfying σ ≤ τ we have hₜ Fₚφ(τ) = Gₚ₁ hₜ. Then h = lim[φ, hₜ] is one-to-one.

Proof. Suppose x and y are points in X such that x ≠ y. Since φ is surjective there is a σ ∈ Σ such that xₚ(φ(σ)) ≠ yₚ(φ(σ)). Since hₜ is one-to-one, hₜ(xₚ(φ(σ))) ≠ hₜ(yₚ(φ(σ))). Thus h(x) ≠ h(y). □

Without the assumption that φ is surjective the previous theorem does not hold as the following example shows.

Example 3.6. Let X₁ = X₃ = {p} and X₂ = [0, 1] with F⁻¹₁(p) = [0, 1], F⁻¹₃(x) = p for every x ∈ [0, 1]. Further, let Y₁ = Y₂ = {p} and G⁻¹₁(p) = p. Finally define φ(1) = 1 and φ(2) = 3. Then (p, 1, p) and (p, 0, p) are in X and h maps each of these to (p, p).

Taken together Theorems 3.1 and 3.5 we have the following corollary.

Corollary 3.7. Let X = lim[Xₙ, F⁻¹ₙ, Λ] and Y = lim[Yₙ, G⁻¹ₙ, Σ] where each of the spaces Xₙ and Yₙ is compact Hausdorff and each of the functions F⁻¹ₙ and G⁻¹ₙ is upper semi-continuous. Further suppose that Σ is a directed set and Λ is linearly ordered, φ : Σ → Λ is an order preserving function such that φ is surjective, hence φ(Σ) is cofinal in Λ, and that, for each σ in Σ, the function hₜ : Xₚ(φ(σ)) → Yₚ is a homeomorphism such that hₜ Fₚφ(τ) = Gₚ₁ hₜ. Then h = lim[φ, hₜ] is a homeomorphism.

In the following example we show inverse systems each having just two factor spaces such that the bonding functions are continuum valued upper semi-continuous functions and there are monotone mappings between the corresponding factor spaces such that the mappings commute but the limit mapping between the inverse limit spaces is not even weakly confluent.

Example 3.8. Let Λ = {1, 2} and let φ : Λ → Λ be the identity function. Define X₁ = [0, 1] × [0, 1] and X₂ = [0, 1]. Let F₁ : X₂ → X₁ be defined as follows. If x ∈ [0, 1/2), F₁(x) is the union of the two line segments the first from the point (x, 0) to the point (1, 0) and the second from (1, 0) to (1, 1). F₁(1/2) is the union of three line segments, the first from (1/2, 0) to (1, 0), the next from (1/2, 0) to (1, 1) and the last from (1, 1) to (1/2, 1). If x ∈ (1/2, 1), then F₁(x) is the union of the two line segments, the first from (1, 0) to (1, 1) and the second from (1, 1) to (x, 1), and finally F₁(1) = X₁. Let Y₁ = Y₂ = [0, 1]. Let h₁ be the projection of X₁ onto its first coordinate and h₂ be the identity. Finally, define G₁ : Y₂ → Y₁ by G₁ = h₁ ◦ F₁.

Let X = lim[Xₙ, F⁻¹ₙ, Λ] and Y = lim[Yₙ, G⁻¹ₙ, Λ], and h = lim[φ, h₁] : X → Y. Then each h₁ is monotone, but we will show that h is not weakly confluent. To this aim define C = {(y₁, y₂) ∈ Y : y₁ ∈ [1/3, 2/3] and y₁ = y₂}. Then C is a subcontinuum of Y and observe that h⁻¹[C] = K₁ ∪ K₂ where K₁ = {[x, y) : x = y and y ∈ [1/3, 1/2]} and K₂ = {((x, 1), y) : x = y and y ∈ [1/2, 2/3]}. Neither of the components K₁ and K₂ are mapped onto C by h.

References