OPEN DIAMETERS ON GRAPHS

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Abstract. We prove that every connected finite graph admits a metric for which the diameter mapping is open.

1. Introduction and Definitions

The main goal of this article is to show that every connected finite graph admits a metric for which the diameter is open.

Diameter is a particular example of a size function. By a size function we mean a real-valued function ω defined on hyperspace of all nonempty compact subset of a set X that satisfies the following two conditions: ω({x}) = 0 for every x ∈ X and if A ⊆ B then ω(A) ≤ ω(B). Other well-known size functions are Whitney maps. They have, by definition, an additional property: if A is a proper subset of B, then ω(A) < ω(B). The subject of openness, monotonicity and confluence of Whitney maps has been investigated by M.M. Awartani, W.J. Charatonik, S.B. Nadler Jr., and A. Illanes, see [1–7].

Much less is known about openness of the diameter mapping. The subject was started by Sam B. Nadler, Jr. in [10, (14.68) Question]. He asked if the circle admits a metric for which the diameter is open. The question was answered positively in [5, Example 5.6]. The article contains some other results, in particular some necessary and some sufficient conditions in order for a continuum X to admit open diameter mappings. It is also shown that every open diameter mapping is monotone, [5, Theorem 3.2].

Here we are continuing the investigation. As mentioned before the main purpose of the article is to show that every finite graph admits a metric for which the diameter mapping is open.

We end the article by posing or recalling some open problems.

Date: November 17, 2016.

2010 Mathematics Subject Classification. 54B20, 54C30, 54D05, 54E40, 54E45, 54F15, 54F50.

Key words and phrases. diameter, finite graph, metric, open diameter.
We consider compact metric spaces only. By a finite graph we mean the finite union of arcs that are either pairwise disjoint or the intersection of any two of them is an endpoint of both together. We also allow a finite number of isolated points. The endpoints of the arcs together with the isolated points are called vertices of the graph and the individual arcs are called edges of the graph. Thus a vertex of a graph may be a ramification point of any finite non-negative order. We denote the set of vertices of the graph $G$ by $V(G)$.

A metric $d$ on a locally connected continuum $X$ is called convex if for every $x, y \in X$ and for every $t \in [0, 1]$ there is a point $z \in X$ such that $d(x, z) = t \cdot d(x, y)$ and $d(y, z) = (1 - t) \cdot d(x, y)$. The existence of convex metrics on locally connected continua was shown by E.E. Moise in [9].

For a given metric continuum $X$, we consider its hyperspace $2^X$ of all its non-empty closed subsets equipped with the Hausdorff distance, or equivalent with the Vietoris topology.

For open subsets $U_1, U_2, \ldots, U_n$ of space $X$ the symbol $\langle U_1, U_2, \ldots, U_n \rangle$ denotes the basic open subset of $2^X$ in the Vietoris topology defined by $\langle U_1, U_2, \ldots, U_n \rangle = \{ A \in 2^X : A \subseteq U_1 \cup \ldots \cup U_n \text{ and } A \cap U_i \neq \emptyset \text{ for } i \in \{1, 2, \ldots n\} \}$.

Let us recall the following definitions from [5].

**Definition 1.1.** A mapping $f : X \to \mathbb{R}$ is lower (upper) semi-open at a point $x \in X$ if for every neighborhood $U$ of $x$ there is a number $\varepsilon > 0$ such that $[f(x) - \varepsilon, f(x)] \cap f(X) \subseteq f(U)$ (or $[f(x), f(x) + \varepsilon] \cap f(X) \subseteq f(U)$, respectively).

Thus a real valued mapping is open if and only if it is both upper and lower semi-open at every point in its domain.

2. General Construction of Metrics

**Definition 2.1.** For a given set $X$, let $R$ be a reflexive, symmetric and transitive subset of $X \times X$. We call a function $d : R \to [0, \infty)$ which satisfies the following conditions

1. $d(x, y) = 0$ if and only if $x = y$ for all $(x, y) \in R$;
2. $d(x, y) = d(y, x)$ for all $(x, y) \in R$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z$ such that $(x, y), (y, z)$ and $(x, z) \in R$.

a partial metric for $X$ with domain $R$.

Let $d_0$ be a metric on $X$ and $d_1, d_2, \ldots, d_n$ be partial metrics for $X$ with domains $R_1, R_2, \ldots, R_n$ respectively and let $R_0 = X \times X$. A path from $x$ to $y$ is the pair $\sigma = ((x_0, \ldots, x_m), k)$ where $k$ is a function
Theorem 2.2. Let $d_0$ be a metric on $X$ and $d_1, d_2, ..., d_n$ be partial metrics for $X$ with domains $R_1, R_2, ..., R_n$ respectively that satisfy the following conditions

1. $R_i$ is a closed subset of $X \times X$ (with the topology on $X$ given by $d_0$);
2. there is $M > 0$ such that $Md_0(x, y) \leq d_i(x, y) \leq d_0(x, y)$ for all $(x, y) \in R_i$.

then the function $d : X \times X \to [0, \infty)$ defined by $d(x, y) = \inf\{\text{len}(\sigma) : \sigma$ is a path from $x$ to $y\}$ is a metric on $X$ equivalent to the original metric $d_0$.

Proof. The conditions $d(x, y) = 0$ if and only if $x = y$ is a consequence of (2). Symmetry is obvious, and the triangle inequality follows from the fact that if $(x_0, x_1, ..., x_n)$ is a path from $x$ to $y$, $(y_0, y_1, ..., y_m)$ is a path from $y$ to $z$, then $(x_0, x_1, ..., x_n, y_1, y_2, ..., y_m)$ is a path from $x$ to $z$. Condition (2) implies that the metrics $d$ and $d_0$ are equivalent. □

As an application of Theorem 2.2 we have the following theorem.

Theorem 2.3. For every nondegenerate continuum $X$ there is a metric $d$ on $X$ such that $\text{diam}_d$ is neither upper nor lower semi-open.

Proof. Let $d_0$ be any metric on $X$. Choose $a, b \in X$, $a \neq b$, and $\varepsilon > 0$ such that $\varepsilon < d_0(a, b)/2$. Define $V = B_{d_0}(a, \varepsilon)$ and $W = B_{d_0}(b, \varepsilon)$. Let $R_1 = (\text{cl}(V) \cup \text{cl}(W)) \times (\text{cl}(V) \cup \text{cl}(W))$ and define

$$d_1(x, y) = \begin{cases} d_0(x, y) & \text{if } x, y \in \text{cl}(V) \text{ or } x, y \in \text{cl}(W) \\ \varepsilon & \text{if } x \in \text{cl}(V) \text{ and } y \in \text{cl}(W) \end{cases}$$

and let $d$ be the metric constructed as in Theorem 2.2 for metrics $d_0$ and $d_1$. Observe that $\text{diam}_d((V, W)) = \{\varepsilon\}$ so $\text{diam}_d$ is not upper nor lower semi-open. □

3. Metrics on connected graphs

Our main goal is to prove the following theorem.

Theorem 3.1. For every connected finite graph $G$ there is a metric $\rho$ on $G$ such that $\text{diam}_\rho : 2^G \to [0, \text{diam}_\rho(G)]$ is open.
To prove the Theorem we will need a series of definitions and results. Let $G$ be a finite connected graph. Let $d$ be a convex metric on $G$ such that $d(x, y) = 1$ for any two adjacent vertices $x, y$ on $G$. Choose a vertex $p \in G$ and define $h : G \to [0, \infty)$ as $h(x) = d(p, x)$. By adding vertices on free arcs we may modify the graph $G$ to get the following additional properties:

(*) If $a$ is a point such that $h(x)$ has local maximum at $a$, then $a$ is a vertex of the modified graph $G$ and $h(a) = \max\{h(x) : x \in G\}$; in particular, if $e$ is an end point of $G$, $e \neq p$ then $h(e) = \max\{h(x) : x \in G\}$.

(**) If $a, b \in V(G)$ then there is at most one edge in $G$ joining $a$ and $b$.

An example of a finite graph and a set of vertices that satisfies conditions (*) and (**) is presented in Figure 1. In this example we have $\max\{h(x) : x \in G\} = 3$ and this maximum is attained at points $f, g, h,$ and $k$.

Figure 1

An arc $A$ in $G$ is said to be monotone if $h|_A$ is monotone. Given points $x$ and $y$ in $G$ the order $x \leq_p y$ means there is a monotone arc with end points $p$ and $y$ that contains $x$.

We will define a new metric $\rho$ on $G$ using Theorem 2.2 in which we will use a metric $hd$ ($hd$ stands for horizontal distance) and only one partial metric $vd$ ($vd$ stands for vertical distance) and show that the diameter mapping for $\rho$ is open.

First, define $hpd : G^2 \to [0, \infty)$ ($hpd$ stands for horizontal projection distance) by $hpd(x, y) = |2^{h(x)+1} - 2^{h(y)+1}|$ and $hd : G^2 \to [0, \infty)$ by $hd(x, y) = \inf\{hpd(x_0, x_1) + hpd(x_1, x_2) + \ldots + hpd(x_{n-1}, x_n) : x = x_0, y = x_n, \text{ and for every } i \in \{1, 2, \ldots, n\} \text{ there is a monotone}$
arc from \(x_{i-1}\) to \(x_i\). Note that \(hd(x, y) = hpd(x, y)\) if \(x \leq_p y\) or \(y \leq_p x\) and \(hd\) is a convex metric on \(G\).

The domain \(R\) for the partial metric \(vd\) is defined by \((x, y) \in R\) if and only if \(h(x) = h(y)\). We will define \(vd\) on \(R\). First, \(vd(x, x) = 0\).

Next, if \(x \neq y\) and \(x, y \in V(G)\) then \(vd(x, y) = 2^{h(x)-1}\).

Continuing, we say \(x\) is related to \(y\) via vertex \(c\) if there are \(a, b, c \in V(G)\) such that \(h(a) = h(b), |h(a) - h(c)| = 1\) and

1. \(c \leq_p x \leq_p a\) and \(c \leq_p y \leq_p b\)

or

2. \(a \leq_p x \leq_p c\) and \(b \leq_p y \leq_p c\)

holds. In this case, we define \(vd(x, y) = \frac{hpd(y, c)}{hpd(c, a)} \cdot vd(a, b)\). Finally, in all other instances, \(vd(x, y) = 2^{h(x)-1}\). Figure 2 shows the graph from Figure 1 with some values of \(hpd(x, y)\) and \(vd(x, y)\).

Let us introduce two new symbols. For a vertex \(c \in V(G)\) define

\[R(c) = \{x \in G : c \leq_p x \text{ and } h(c) \leq h(x) \leq h(c) + 1\} \text{ and } L(c) = \{x \in G : x \leq_p c \text{ and } h(c) - 1 \leq h(x) \leq h(c)\}\]

thus \(x\) and \(y\) are related via vertex \(c\) if and only if \(x, y \in R(c)\) or \(x, y \in L(c)\).

**Observation 3.2.** If \(x, y \in R(c)\), \(x \neq y\), and \(h(x) = h(y)\) then \(vd(x, y) = hpd(x, c)/2\). Similarly, if \(x, y \in L(c)\) and \(h(x) = h(y)\) then \(vd(x, y) = hpd(x, c)/4\).

Note that if \(c, d \in V(G)\) and \(c \neq d\), then the sets \(R(c) \cap R(d)\) and \(L(c) \cap L(d)\) are contained in \(V(G)\).

**Proposition 3.3.** The function \(vd\) is a partial metric with domain \(R\).

**Proof.** The reflexivity and symmetry are obvious, we need to check only the triangle inequality. To this aim, first observe that in all cases we have \(vd(x, z) \leq 2^{h(x)-1}\). If \(vd(x, y) = 2^{h(x)-1}\) or \(vd(y, z) = 2^{h(x)-1}\), then \(vd(x, z) \leq vd(x, y) + vd(y, z)\) follows from the previous observation. If all of \(vd(x, y), vd(x, z),\) and \(vd(y, z)\) are defined using condition (1) or all are defined using condition (2), then the triangle \(a, b, c\) is equilateral, so the conclusion follows. For the remaining case assume \(vd(x, y)\) is defined using condition (1) and \(vd(y, z)\) is defined using condition (2).

Let \(a, b, c, d \in V(G)\) such that \(a, b, c\) satisfy condition (1) for \(vd(x, y)\) and \(c, d, b\) satisfy condition (2), with \(h(d) = h(c) = h(b) - 1\), for \(vd(y, z)\).
Then
\[ vd(x, y) + vd(y, z) = \frac{hpd(c, x)}{hpd(c, b)} vd(a, b) + \frac{hpd(y, b)}{hpd(c, b)} \vdots \frac{vd(a, b)}{2} \]
\[ = \frac{hpd(c, x)}{hpd(c, b)} (hpd(c, x) + \frac{1}{2} hpd(y, b)) \]
\[ = \frac{vhd(a, b)}{hpd(c, b)} \left( hpd(c, x) + \frac{1}{2} (hpd(c, b) - hpd(c, x)) \right) \]
\[ = \frac{vhd(a, b)}{hpd(c, b)} \frac{1}{2} (hpd(c, x) + hpd(c, b)) \]
\[ = \frac{1}{2} (2^{h \alpha} + 1 - 2^{h(b)+1} + 2^{h(b)+1} - 2^{h(c)+1} \right) \]
\[ = \frac{1}{2} \left( \frac{2^{h(x)+1} - 2^{h(c)+1} + 2^{h(c)+2} - 2^{h(c)+1} \right) \]
\[ = \frac{1}{4} (2^{h(x)+1}) = 2^{h(x)-1}. \]

But as observed earlier, \( vd(x, z) \leq 2^{h(x)-1} \) thus the inequality holds.

An alternate way to see why equality is obtain in the last case is to notice that the quantity \( vd(x, y) + vd(y, z) - vd(x, z) \) as \( x \) moves from \( c \) to \( a \) is a linear function, \( F(\alpha) \), of one independent variable \( \alpha = hpd(c, x) \). If \( \alpha = 0 \), \( x = c \) and \( F(\alpha) = 0 \). If \( \alpha = hpd(c, a) \), \( x = a \) and \( F(\alpha) = 0 \). So \( F(\alpha) = 0 \) for all \( \alpha \in [0, hpd(c, a)] \).

**Definition 3.4.** Let \( \rho \) be a metric on \( G \) defined as in the conclusion of Theorem 2.2 for the metric \( h d \) in place of \( d_0 \) and the partial metric \( v d \) in place of the (only) partial metric \( d_1 \).

We next develop some useful properties of the metric \( \rho \).

**Proposition 3.5.** For any points \( x, y \in G \), we have \( \rho(x, y) \geq hpd(x, y) \).
Proposition 3.6. If \( (x_0, x_1, \ldots, x_n, k) \) be a path from \( x \) to \( y \), and \( x_{i-1}, x_i \in \{x_0, x_1, \ldots, x_n\} \). If \( x_i \leq_p x_{i+1} \) or \( x_{i+1} \leq_p x_i \) then \( d_{k(i)}(x_i, x_{i+1}) = hpd(x_i, x_{i+1}) \) otherwise either

(a) there is \( c \in V(G) \) such that \( d_{k(i)}(x_i, x_{i+1}) = hpd(x_i, c) + hpd(c, x_{i+1}) > hpd(x_i, x_{i+1}) \)

or

(b) \( d_{k(i)}(x_i, x_{i+1}) = v d(x_i, x_{i+1}) > hpd(x_i, x_{i+1}) = 0 \). So

\[
\sum_{i=0}^{n-1} d_{k(i)}(x_i, x_{i+1}) \geq \sum_{i=0}^{n-1} hpd(x_i, x_{i+1}) \geq hpd(x, y). \]

Hence \( \rho(x, y) \geq hpd(x, y) \) as needed.

\[\square\]

Proposition 3.6. If \( x \) and \( y \) are points in \( G \) such that \( x \leq_p y \), then \( \rho(x, y) = hd(x, y) \).

Proof. The inequality \( \rho(x, y) \geq hd(x, y) \) is a consequence of Proposition 3.5. To see that \( \rho(x, y) \leq hd(x, y) \) it is enough to take the path \( \sigma = (x, y, k) \) in the definition of \( \rho \).

\[\square\]

Lemma 3.7. There is a number \( N \) such that for any path \( \sigma = ((x_0, \ldots, x_n), k) \) from \( x \) to \( y \) there is a path \( \sigma' = ((y_0, \ldots, y_m), k') \) from \( x \) to \( y \) such that

1. \( \text{len}(\sigma') \leq \text{len}(\sigma) \);
2. \( k'(i) \neq k'(i + 1) \) for \( i \in \{0, \ldots, m - 1\} \);
3. If \( m > 1 \), then for any monotone arc \( I \) there are at most two points in \( \{y_0, y_1, \ldots, y_m\} \cap I \);
4. \( m = N \).

Proof. First, if there is a monotone arc \( I \) such that \( x_i, x_j \in I \) and \( i < j \), then replacing the sequence \( x_i, x_{i+1}, \ldots, x_j \) by \( x_i, x_j \), we get a new path \( \sigma_1 = ((x_0, x_1, \ldots, x_i, x_j, \ldots, x_n), k_1) \) from \( x \) to \( y \), by Proposition 3.6, satisfying \( \text{len}(\sigma_1) \leq \text{len}(\sigma) \).

Second, if \( k_1(i) = k_1(i+1) \), then we may replace the points \( x_i, x_{i+1}, x_{i+2} \) by \( x_i, x_{i+2} \) only and by this change we obtain a new path \( \sigma_2 \) from \( x \) to \( y \) satisfying \( \text{len}(\sigma_2) \leq \text{len}(\sigma_1) \leq \text{len}(\sigma) \) (because of the triangle inequality for \( d_{k(i)} \)). After this modification, made for all indices \( i \) such that \( k_1(i) = k_1(i + 1) \), we get a new path \( \sigma_3 \) that satisfies conditions (1)-(3).

Conditions (2), (3) and the fact that there is a finite number of maximal monotone arcs in \( G \) gives an upper bound for the numbers \( m \). Let \( N \) be the maximum of those numbers. To get condition (4) it is enough to append to the sequence \( x_0, x_1, \ldots, x_m \) the constant sequence \( y, y, \ldots, y \) of length \( N - m \) and define \( k' : \{1, \ldots, N\} \) by \( k'(i) = k_3(i) \) for \( i \leq m \) and such that \( k' \) satisfies (2).

\[\square\]
Theorem 3.8. For any points \( x, y \in G \) there is a path \( \sigma \) from \( x \) to \( y \) such that \( \rho(x, y) = \text{len}(\sigma) \).

Proof. Let \( N \) be the number guaranteed by Lemma 3.7 and let \( \sigma_i = ((x_{i,0}, \ldots, x_{i,N}), k_i) \) be a sequence of paths between \( x \) and \( y \) such that \( \rho(x, y) = \lim_{i \to \infty} \text{len}(\sigma_i) \). By compactness of \( G \) there is a sequence \( \{i_n\}_{n=1}^{\infty} \) such that \( (x_{i_n,0}, \ldots, x_{i_n,N}) \in G^N \) is convergent to a point \((x_0, x_1, \ldots, x_N)\) and that \( k_{i_n} = k_{i_m} \) for all \( m, n \). Then \( \sigma = ((x_0, x_1, \ldots, x_N), k_{i_n}) \) is the required path. \( \square \)

Corollary 3.9. There is a number \( N \) such that for any points \( x, y \in G \) there is a path \( \sigma = ((x_0, \ldots, x_n), k) \) from \( x \) to \( y \) such that

1. \( \rho(x, y) = \text{len}(\sigma) \);
2. \( k(i) \neq k(i + 1) \) for \( i \in \{0, \ldots, n - 1\} \);
3. \( x_i \neq x_{i+1} \) for \( i \in \{0, \ldots, n - 1\} \);
4. For any monotone arc \( I \) there are at most two points in \( \{x_0, x_1, \ldots, x_n\} \cap I \);
5. \( n \leq N \).

Proof. By Theorem 3.8 there is a path satisfying (1), using Lemma 3.7, we may modify it to a path that satisfies conditions (1),(2),(4) and (5). Eliminating repeating points we can get a new path that satisfies all the required conditions. \( \square \)

Proposition 3.10. If \( x, y \in G \) with \( h(x) = h(y) \) and \( x \neq y \) then \( \text{vd}(x, y) < \text{hd}(x, y) \).

Proof. If \( x \) and \( y \) are related via a vertex \( c \) let \( a, b \in V(G) \) such that \( a \) and \( b \) are related via vertex \( c \) and \( x \) and \( y \) are on monotone arcs from \( c \) and \( a \) and \( c \) to \( b \) respectively. Then \( \text{vd}(x, y) = \frac{\text{hpd}(x,c)}{\text{hpd}(a,c)} \cdot \text{vd}(a,b) \leq \text{hpd}(x,c) \) while \( \text{hd}(x, y) = 2\text{hpd}(x,c) \) so \( \text{vd}(x, y) < \text{hd}(x, y) \) in this case.

If \( x \) and \( y \) are not related via a vertex then \( \text{vd}(x, y) = 2^{h(x)-1} \). Let \( c \in V(G) \) be such that there is a monotone arc from \( c \) to \( x \) and a monotone arc from \( c \) to \( y \) and \( |\text{hpd}(c) - \text{hpd}(x)| \) is minimum. Then \( |h(c) - h(x)| \geq 1 \) and

\[
\text{hd}(x, y) = 2\text{hpd}(x,c) = 2^{2^{h(x)+1} - 2^{h(c)+1}} \\
= 2 \cdot 2^{h(x)+1} |1 - 2^{h(c) - h(x)}| \\
\geq 8 \cdot 2^{h(x)-1} \cdot \frac{1}{2} \\
= 4\text{vd}(x, y) \\
> \text{vd}(x, y).
\]

\( \square \)
Lemma 3.11. If \( x \) and \( y \) are points of \( G \) and there is no monotone arc from \( x \) to \( y \) then \( \rho(x,y) < \text{hd}(x,y) \).

Proof. If \( h(x) = h(y) \) the conclusion follows from Proposition 3.10 so we may assume that \( h(x) < h(y) \). Let \( A \) be a convex arc from \( x \) to \( y \) so the length of \( A = \text{hd}(x,y) \). Since \( x \) and \( y \) are not on a monotone arc, the function \( h : A \to h(A) \) is not a homeomorphism. Thus there exists distinct points \( c_1, c_2 \in A \) such that \( h(c_1) = h(c_2) \) and \( c_1 \) precedes \( c_2 \) in the order from \( x \) to \( y \). Then \( \text{hd}(x,y) = \text{hd}(x,c_1) + \text{hd}(c_1,c_2) + \text{hd}(c_2,y) > \text{hd}(x,c_1) + \text{vd}(c_1,c_2) + \text{hd}(c_2,y) \) by Proposition 3.10. The last quantity is greater than or equal to \( \rho(x,y) \). \( \square \)

Lemma 3.12. If \( x, y, x', y' \in G \) such that \( h(x) = h(y) \neq h(x') = h(y') \), there is a natural number \( n \) such that \( h(x), h(x') \in [n, n+1] \), the points \( x \) and \( x' \) lie on a monotone arc, the points \( y \) and \( y' \) lie on a monotone arc, and \( \sigma = ((x, x', y', y), k) \) is a path from \( x \) to \( y \) then \( \text{vd}(x,y) < \text{len}(\sigma) \).

Proof. It follows from Proposition 3.10 that we may suppose \( \text{len}(\sigma) = \text{hpd}(x,x') + \text{vd}(x',y') + \text{hpd}(y',y) \).

Case (1): The points \( x \) and \( y \) are not related via a vertex.

Case (1.1): \( h(x) < h(x') \). Then \( \text{vd}(x,y) = 2^{h(x)-1} \). Since \( h(x), h(x') \in [n, n+1] \) it follows that \( \text{vd}(x',y') = 2^{h(x')-1} \) so \( \text{vd}(x',y') > \text{vd}(x,y) \). Then \( \text{len}(\sigma) > \text{vd}(x',y') > \text{vd}(x,y) \).

Case (1.2) \( h(x) > h(x') \). An easy computation shows that \( \text{len}(\sigma) - \text{vd}(x,y) = 7(2^{h(x)-1} - 2^{h(x')-1}) > 0 \).

Case (2): The points \( x \) and \( y \) are related via a vertex \( c \). Then \( x' \) and \( y' \) are also related via vertex \( c \). Let \( a, b \in V(G) \) such that \( h(a) = h(b) \), \( |h(c) - h(a)| = 1 \), and \( x, y \) lie on monotone arcs from \( c \) to \( a \) and \( c \) to \( b \) respectively. Then

\[
\text{len}(\sigma) - \text{vd}(x,y) = \text{hpd}(x,x') + \text{vd}(x',y') + \text{hpd}(y',y) - \text{vd}(x,y) \\
= 2\text{hpd}(y,y') + \frac{\text{hpd}(y',y)}{\text{hpd}(c,a)}\text{vd}(a,b) - \frac{\text{hpd}(y',c)}{\text{hpd}(c,a)}\text{vd}(a,b) \\
= 2\text{hpd}(y,y') + \frac{\text{hpd}(c,a)}{\text{hpd}(c,a)}\text{vd}(a,b)(\text{hpd}(y',c) - \text{hpd}(y,c)) \\
= 2\text{hpd}(y,y') + \frac{\text{hpd}(c,a)}{\text{hpd}(c,a)}((\pm)\text{hpd}(y,y')) \\
= \text{hpd}(y,y')(2 \pm \frac{\text{hpd}(a,b)}{\text{hpd}(c,a)}) \\
\geq \text{hpd}(y,y')(2 - \frac{1}{2}) > 0.
\]

Note that the \((\pm)\) sign depends on whether \( h(c) \geq h(y) \) or not. \( \square \)

Lemma 3.13. If \( x, y, x', y' \in G \) such that \( h(x) = h(y) \), \( h(x') = h(y') \), and \( \sigma = ((x, x', y', y), k) \) is a path from \( x \) to \( y \) then \( \text{vd}(x,y) \leq \text{len}(\sigma) \). Moreover, if \( h(x) \neq h(x') \), then \( \text{vd}(x,y) < \text{len}(\sigma) \).
Proof. Case (1) \( h(x) = h(x') \), and hence \( h(y) = h(y') \). It follows from Proposition 3.10 that \( \text{len}(\sigma) \geq vd(x, x') + vd(x', y') + vd(y', y) \) and the conclusion follows by the triangle inequality for the \( vd \) partial metric.

Case (2) \( h(x) \neq h(x') \). Then we must use the \( hd \) metric for \( x \) to \( x' \) and \( y' \) to \( y \). It again follows from Proposition 3.10 and the definition of the \( hd \) metric that \( hd(x, x') + hd(x', y') + hd(y', y) \geq hpd(x, x') + vd(x', y') + hpd(y', y) \). So we will show that \( vd(x, y) \) is less than or equal to this last quantity.

Case (2.1) \( |h(x) - h(x')| > 1 \). Then \( hpd(x, x') = |2^{h(x)+1} - 2^{h(x')} + 1| = 2^{h(x)-1} - 2^{h(x') - h(x) + 2} \geq vd(x, y) \cdot 2 \) and thus \( vd(x, y) < \text{len}(\sigma) \).

Case (2.2a) \( |h(x) - h(x')| \leq 1 \) and there is no positive integer \( n \) such that \( h(x) \leq n \leq h(x') \). We may chose points \( x'' \) and \( y'' \) so that \( h(x) = h(x'') \), \( h(y) = h(y'') \), \( x \) and \( x'' \) lie on the same monotone arc, and \( y \) and \( y'' \) lie on the same monotone arc. Note that \( hpd(x, x') = hpd(x'', x') = hd(x'', x') \) and similarly for the points \( y, y', \) and \( y'' \). Then the conclusion follows from applying Lemma 3.12 to the path \( \{ (x'', x', y', y'') \} \).

Case (2.2b) \( |h(x) - h(x')| \leq 1 \) and there is a positive integer \( n \) such that \( h(x) < n < h(x') \). Let \( a, b \in V(\mathcal{G}) \) with \( h(a) = h(b) = n \), \( a \) and \( x' \) on the same monotone arc, \( b \) and \( y' \) on the same monotone arc. Then \( hpd(x, x') + vd(x', y') + hpd(y', y) = hpd(x, a) + hpd(a, x') + vd(x', y') + hpd(y', b) + hpd(b, y) \geq hpd(x, a) + vd(a, b) + hpd(b, y) \) by Proposition 3.12. Finally, if we replace the points \( a \) and \( b \) with points \( a' \) and \( b' \) which have the same \( h \) value and are on monotone arcs from \( x \) and \( y \) respectively, we can apply Lemma 3.12 again and obtain the desired conclusion.

Proposition 3.14. If \( x, y \in \mathcal{G} \) with \( h(x) = h(y) \) then \( vd(x, y) = \rho(x, y) \). Moreover, if \( \sigma \) is any path \( x \) to \( y \) that contains a point other than \( x \) and \( y \), then \( \text{len}(\sigma) > \rho(x, y) \).

Proof. Suppose \( \sigma = ((x_0, x_1, \ldots, x_m), k) \) is a path from \( x \) to \( y \) with \( m \geq 2 \) that satisfies the conclusion of Corollary 3.9. By condition (2) of Corollary 3.9 there is \( j, 1 \leq j \leq m - 1 \) such that \( d_{k(j)}(x_{j-1}, x_j) = hd(x_{j-1}, x_j) \). It follows from Proposition 3.10 that \( h(x_{j-1}) \neq h(x_j) \). So for some \( j \), \( h(x_j) \neq h(x) \). Assume \( h(x_j) < h(x) \) and let \( x_i \in \{ x_0, x_1, \ldots, x_m \} \) such that \( h(x_i) \leq h(x_k) \), for all \( k \in \{ 0, 1, \ldots, m \} \). Again, by condition (2) of Corollary 3.9 either (a) \( h(x_{i-1}) = h(x_i) \) or (b) \( h(x_i) = h(x_{i+1}) \). Assume (a) holds. If \( x_i \leq p x_{i-1} \) then there exists \( x_{i-2} \) such that \( h(x_{i-2}) = h(x'_{i-2}) \) and \( x_i \leq p x_{i-2} \). But \( hd(x_{i-2}, x_{i-1}) > vd(x_{i-2}, x_{i-2}) + hd(x_{i-2}, x_i) \). So \( \sigma \) could be modified.
to a path from $x$ to $y$ having length less than $\rho(x, y)$, a contradiction. Thus $x_{i-1} \leq_p x_{i-2}$. Likewise, $x_i \leq_p x_{i+1}$.

Next, if $h(x_{i+1}) < h(x_{i-2})$ let $x_{i+1}'$ be such that $x_{i-1} \leq_p x_{i+1}'$ and $h(x_{i+1}') = h(x_{i+1})$. Then by Lemma 3.13, $hd(x_{i-2}, x_{i+1}') + vd(x_{i+1}', x_{i+1}) < hd(x_{i-2}, x_{i-1}) + vd(x_{i-1}, x_i) + hd(x_i, x_{i+1})$ again giving a path from $x$ to $y$ with length less than $\rho(x, y)$.

The other possible cases can be handled similarly. Thus $m = 1$ and $vd(x, y) = \rho(x, y)$. \hfill \Box

**Remark 3.15.** As a consequence of Proposition 3.14, if $\sigma = ((x_0, \ldots, x_n), k)$ is a path from $x$ to $y$ such that $\text{len}(\sigma) = \rho(x, y)$ the use of either $hd$ or $vd$ in $k$ can be determined by the points $x_0, \ldots, x_n$, namely, if $h(x_i) \neq h(x_{i+1})$ we have to use $hd$, if $h(x_i) = h(x_{i+1})$ we use $vd$; therefore we will omit the function $k$ is the description of $\sigma$ in this case.

**Proposition 3.16.** There is a number $N$ such that for any points $x, y \in G$ satisfying $h(x) \leq h(y)$ there is a path $\sigma = ((x_0, \ldots, x_n), k)$ from $x$ to $y$ such that

1. $\rho(x, y) = \text{len}(\sigma)$;
2. $k(i) \neq k(i + 1)$ for $i \in \{0, \ldots, n - 1\}$;
3. $x_i \neq x_{i+1}$ for $i \in \{0, \ldots, n - 1\}$;
4. for any monotone arc $I$ there are at most two points in $\{x_0, x_1, \ldots, x_n\} \cap I$;
5. $n \leq N$;
6. $h(x_0) \leq h(x_1) \leq \ldots \leq h(x_n)$;
7. if $h(x_i) \neq h(x_{i+1})$, then $x_i \leq_p x_{i+1}$ and thus $\rho(x_i, x_{i+1}) = hd(x_i, x_{i+1})$.

**Proof.** Let $\sigma$ be any path that satisfies Corollary 3.9 we will show that $\sigma$ satisfies a conditions (6) and (7). First, observe the following: by Lemma 3.11, every time $k(j)$ corresponds to using the $hd$ metric to calculate the distance from $x_j$ to $x_{j+1}$ we must have a monotone arc from $x_j$ to $x_{j+1}$.

Suppose (6) is not satisfied, then there is an index $i$ such that one of the following two cases holds:

**Case 1:** $h(x_{i-1}) < h(x_i) = h(x_{i+1}) > h(x_{i+2})$

**Case 2:** $h(x_{i-1}) > h(x_i) = h(x_{i+1}) < h(x_{i+2})$

In each of Case (1) and Case (2) we need to consider two subcases.

**Case 1a:** Case (1) and $h(x_{i-1}) \leq h(x_{i+2})$

**Case 1b:** Case (1) and $h(x_{i-1}) \geq h(x_{i+2})$

**Case 2a:** Case (2) and $h(x_{i-1}) \leq h(x_{i+2})$

**Case 2b:** Case (2) and $h(x_{i-1}) \geq h(x_{i+2})$
Since all cases are very similar, we will show how to proceed in one of them, say Case (2a). Let \( z \in G \) be a point such that \( x_{i+1} \leq_p z \leq_p x_{i+2} \) and \( h(x_{i-1}) = h(z) \). Define \( \sigma = (x_0, x_1, \ldots, x_{i+1}, z, x_{i+2}, \ldots, x_n) \) and observe that \( \text{len}(\sigma') = \text{len}(\sigma) \). Continuing, define \( \sigma'' = (x_0, x_1, \ldots, x_{i-1}, z, x_{i+2}, \ldots, x_n) \) and note that, by Proposition 3.14 \( \text{len}(\sigma'') < \text{len}(\sigma') = \text{len}(\sigma) \) contrary to condition (1). This finishes the proof of condition (6).

Condition (7) follows from (6) and the observation at the beginning of the proof.

**Lemma 3.17.** If \( x, x', y, y' \in G \) with \( h(x) = h(y) \neq h(x') = h(y') \) and \( x \) and \( x' \) are on a monotone arc and \( y \) and \( y' \) are on a monotone arc then \( vd(x, y) + \text{hpd}(y, y') > vd(x', y') \).

**Proof.** Define \( F(x) = vd(x, y) + \text{hpd}(y, y') - vd(x', y') \). We wish to show that \( F(x) > 0 \) for all \( x \in G \). Once again we need to consider several cases.

**Case 1:** \( x \in V(G) \) and \( h(x) < h(x') \).

**Case 1a:** \( y \neq x \); \( F(x) = 2^{h(x)-1} + 2^{h(y)+1} - 2^{h(y)+1} - vd(x', y') \geq 2^{h(x)-1} + 2^{h(y)+1} - 2^{h(y)+1} = 3(2^{h(y)-1}) > 0 \).

**Case 1b:** \( y = x \).

**Case 1bi:** \( y' \notin R(x) \). \( F(x) > 0 \) follows from Observation 3.2.

**Case 1bii:** \( y' \notin R(x) \). Then \( h(x') > h(x) + 1 \) and \( F(x) = 2^{h(x')+1} - 2^{h(x)+1} - 2^{h(x')-1} = 3(2^{h(x)-1}) - 2^{h(x)+1} > 2^{h(x)} \cdot 3 - 2^{h(x)} \cdot 2 > 0 \).

**Case 2:** \( x \in V(G) \) and \( h(x) > h(x') \). Similar calculations handle all of the various subcases.

**Case 3:** \( x \notin V(G) \). The functions \( vd \) and \( \text{hpd} \) are linear functions of the variable \( 2^{h(x)} \) for \( h(x) \in [n, n + 1] \). Thus \( F \) is a linear function of the variable \( 2^{h(x)} \) when \( h(x) \in [n, n + 1] \) and is positive when \( h(x) \) is an integer. Thus \( F(x) > 0 \) for all \( x \in G \).

**Corollary 3.18.** If \( a, b, c \in V(G), x, x', y, y' \in G \) such that \( a, b, c, x, x', y, y' \in R(c), x' \leq_p x, y' \leq_p y, h(x') < h(x), h(x) = h(y), h(x') = h(y') \) then \( \rho(x, y) = vd(x', y') + \text{hpd}(y, y') > vd(x, y) = \rho(x, y) \).

**Corollary 3.19.** If \( a, b, c \in V(G), x, x', y, y' \in G \) such that \( a, b, c, x, x', y, y' \in V(c), x' \leq_p x, y' \leq_p y, h(x') < h(x), h(x) = h(y), h(x') = h(y') \) then \( \rho(x, y) = vd(x, y) + \text{hpd}(y, y') > vd(x', y') = \rho(x', y') \).

**Proposition 3.20.** For any \( x, y \in G \) with \( h(x) \leq h(y) \) there is a path \( \sigma = (x, x_1, x_2, y) \) such that:

1. \( \rho(x, y) = \text{len}(\sigma) \);
2. \( h(x) \leq h(x_1) \leq h(x_2) \leq h(y) \);
(3) \( x_1 \leq_p x_2 \);
(4) \( \rho(x, y) = vd(x, x_1) + hpd(x_1, x_2) + vd(x_2, y) \).

**Proof.** Given two points \( x, y \in G \) with \( h(x) \leq h(y) \), let \( \sigma = (x_0, \ldots, x_N) \) be a path from \( x \) to \( y \) satisfying the conditions (1)-(7) of Proposition 3.16.

First we will show that \( N \leq 3 \). Suppose on the contrary that \( N \geq 4 \). Then there are points \( x_i, x_{i+1}, x_{i+2}, x_{i+3} \) such that \( \rho(x_i, x_{i+3}) = hpd(x_i, x_{i+1}) + vd(x_{i+1}, x_{i+2}) + hpd(x_{i+2}, x_{i+3}) \). Here we have \( h(x_i) < h(x_{i+1}) = h(x_{i+2}) < h(x_{i+3}) \). Consider the following cases:

Case 1: The points \( x_{i+1}, x_{i+2} \in R(c) \) for some vertex \( c \) but \( x_{i+1}, x_{i+2} \notin V(G) \).

Case 1a: \( h(x_i) \leq h(c) \). Because \( x_{i+1} \) is not a vertex, we have \( x_i \leq_p c \leq_p x_{i+2} \), and the path \( \sigma' = (x_0, \ldots, x_{i+2}, x_{i+3}, \ldots, x_N) \) satisfies \( \text{len}(\sigma') < \text{len}(\sigma) \) contrary to (1) of Proposition 3.16.

Case 1b: \( h(x_i) \geq h(c) \). Let \( z \) be the points such that \( c \leq_p z \leq_p x_{i+2} \) and \( h(z) = h(x_i) \) by Corollary 3.18 we have \( \rho(x_i, x_{i+2}) = vd(x_i, z) + hpd(z, x_{i+2}) \) so the path \( \sigma' = (x_0, \ldots, x_i, z, x_{i+2}, \ldots, x_N) \) is shorter than the original path \( \sigma \), a contradiction.

Case 2: The points \( x_{i+1}, x_{i+2} \in L(d) \) for some vertex \( d \) but \( x_{i+1}, x_{i+2} \notin V(G) \).

Case 2a: \( h(x_{i+3}) < h(d) \). We proceed similarly as in Case 1b. Let \( z \) be the points such that \( x_{i+1} \leq_p z \leq_p d \) and \( h(z) = h(x_{i+3}) \). Then by Corollary 3.19 the path \( (x_0, \ldots, x_{i+1}, z, x_{i+3}, \ldots, x_N) \) is shorter than \( \sigma \).

Case 2b: \( h(x_{i+3}) \geq h(d) \). Because \( x_{i+2} \) is not a vertex, we have \( x_{i+1} \leq_p d \leq_p x_{i+3} \) and the path \( (x_0, \ldots, x_{i+1}, x_{i+3}, \ldots, x_N) \) is shorter than the original path \( \sigma \).

Case 3: The points \( x_{i+1} \) and \( x_{i+2} \) are vertices or they are not related via any vertex. Let \( z \) and \( w \) be a points satisfying \( x_i \leq_p z \leq_p x_{i+1} \), \( w \leq_p x_{i+2} \), and \( h(z) = h(w) < h(x_{i+1}) = h(x_{i+2}) \). Then \( vd(x_{i+1}, x_{i+2}) = 2^{h(x_{i+1})-1} > 2^{h(z)-1} \geq vd(z, w) \) and then the path \( (x_0, \ldots, x_i, z, w, \ldots, x_N) \) is shorter than the original path, a contradiction.

This exhausts all the cases and finishes the proof that \( N \leq 3 \) and if \( N = 3 \), then \( \rho(x, y) = vd(x, x_1) + hpd(x_1, x_2) + vd(x_2, y) \).

If \( N < 3 \), it is enough to add constants to the path \( \sigma \), for example if \( x \leq_p y \), then the path \( (x, x, y, y) \) satisfies the conclusion.

\( \square \)

**Example 3.21.** To see that \( N = 3 \) may be necessary consider the case that \( c \) and \( d \) are vertices with \( h(d) = h(c) + 1 \) and there is a monotone arc from \( c \) to \( d \). Further, that \( x \in R(c) \) and there is no monotone arc from \( x \) to \( d \), \( y \in L(d) \) and there is no monotone arc
from $y$ to $c$. If $h(x) - h(c)$ and $h(d) - h(y)$ are small enough so that
$$vd(x, x_1) + vd(x_2, y) < 2h(x) - 1$$
where $x_1$ and $x_2$ are on the monotone arc from $c$ to $d$ with $h(x) = h(x_1)$ and $h(y) = h(x_2)$ then path $(x, x_1, x_2, y)$ has length less than $(x, z, y)$ where $h(x) = h(z)$ and $z$ is on a monotone arc with $y$ or $h(y) = h(z)$ and $z$ is on a monotone arc with $x$.

**Lemma 3.22.** If $a, b, c, x \in G$ such that $h(a) \leq h(b)$ and $c \leq_p a$, $c \neq a$, then $\rho(c, b) > \rho(a, b)$.

**Proof.** It follows from $c \leq_p a$ and $c \neq a$ that $h(c) < h(b)$. Let $\sigma = (c, x_1, x_2, b)$ such that $\rho(c, b) = \text{len}(\sigma)$ given by Lemma 3.20. Then there are two cases.

**Case 1:** $c \neq x_1$. If follows from Lemma 3.20 that $h(c) = h(x_1)$. Thus there is a $z$ such that $z \leq_p x_2$ and $h(z) = h(a)$. Then by Lemma 3.17 $vd(c, x_1) + hpd(x_1, z) > vd(a, z)$ so $\rho(c, b) > \text{len}(a, z, x_2, b) \geq \rho(a, b)$.

**Case 2:** $c = x_1$. In this case $hpd(c, x_2) = hpd(c, a) + hpd(a, x_2)$. So again $\rho(c, b) > \text{len}(a, x_2, b) \geq \rho(a, b)$.

The proof of the following lemma is analogous to the proof of the previous one.

**Lemma 3.23.** If $a, b, c \in G$ such that $h(a) \leq h(b)$ and $b \leq_p c$, $c \neq b$, then $\rho(a, c) > \rho(a, b)$.

Define $M = \max\{h(x) : x \in G\}$. As a consequence of Proposition 3.6 and Lemmas 3.22 and 3.23, we have following Corollary.

**Corollary 3.24.** $\text{diam}_\rho(G) = 2^{M+1} - 2$.

**Proposition 3.25.** The diameter mapping $\text{diam}_\rho$ is upper semi-open.

**Proof.** Let $\delta > 0$ and let $a$ and $b$ be two points of a set $A$ such that $\rho(a, b) = \text{diam}_\rho(A) < \text{diam}_\rho(G)$ and $h(a) \leq h(b)$. Then, by Corollary 3.24, either $h(a) > 0$ or $h(b) < M$.

If $h(a) > 0$, let $a'$ be a point satisfying $a' \leq_p a$ and $\rho(a, a') < \delta$, and let $I$ be a monotone arc with end-points $a$ and $a'$. Then $A = \{A \cup \{x\} : x \in I\}$ is contained in the $\delta$-ball in $2^G$ centered at $A$. Furthermore, observe that, for $x \in I$, we have $\rho(x, b) > \rho(a, b)$ by Lemma 3.22, so $\text{diam}_\rho(A)$ is a continuous image of the arc $A$, so it is of the form $[\text{diam}_\rho(A), \text{diam}_\rho(A) + \varepsilon]$ for some $\varepsilon > 0$.

Similarly, if $h(b) < M$, let $b'$ be a point satisfying $b \leq_p b'$ and let $J$ be a monotone arc with end-points $b$ and $b'$. Then $B = \{A \cup \{x\} : x \in J\}$ is contained in the $\delta$-ball in $2^G$ centered at $A$. Arguing similarly as in the previous case, using Lemma 3.23, one can show that $\text{diam}_\rho(B)$ is of the form $[\text{diam}_\rho(A), \text{diam}_\rho(A) + \varepsilon]$ for some $\varepsilon > 0$. □
For a compact set \( A \), define \( M(A) \subseteq A \times A \) by \( M(A) = \{(x, y) \in A \times A : \rho(x, y) = \text{diam}_{\rho}(x, y)\} \). Thus \( M(A) \) is a nonempty compact subset of \( A \times A \). For a vertex \( v \in V(G) \) define \( \text{st}(v) = \{y \in G : y \in R(v) \cup L(v)\} \). In other words, \( \text{st}(v) \) is the union of all edges that contain the vertex \( v \).

**Lemma 3.26.** If \( v \in V(G) \) and \( A \subseteq \text{st}(v) \), then \( \text{diam}_\rho \) is lower semi-open at \( A \).

**Proof.** Define \( H : \text{st}(v) \times [0, 1] \to \text{st}(v) \) by the condition \( y = H(x, t) \) if and only if \( x \leq_p y \leq_p v \) or \( v \leq_p y \leq_p x \) and \( \frac{hpd(y, v)}{hpd(x, v)} = 1 - t \).

Note that \( H(x, 0) = x \), \( H(x, 1) = v \), and if \( x \neq y \) and \( t > 0 \), then \( \rho(H(x, t), H(y, t)) < \rho(x, y) \). Therefore, for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \( H(A \times [0, \delta]) \) is contained in the neighborhood of \( A \) radius \( \varepsilon \). Moreover, if \( A \) is non-degenerate \( \text{diam}_\rho(H(A \times [0, \delta])) \) is of the form \( [\text{diam}_\rho(A) - \eta, \text{diam}_\rho(A)] \) for some \( \eta > 0 \) as required. \( \Box \)

**Definition 3.27.** For a set \( A \subseteq 2^G \) define \( b(A) \) as the least integer such that \( h(x) \leq b(A) + 1 \) for every \( x \in A \). In particular, there is a point \( a \in A \) such that \( b(A) \leq h(a) \leq b(A) + 1 \). Notice that \( b(A) < \text{max}\{h(x) : x \in A\} \leq M \).

**Lemma 3.28.** If \( \text{diam}_\rho(A) < 2^{b(A) - 1} \), then there is a vertex \( v \in G \) such that \( A \subseteq \text{st}(v) \).

**Proof.** Consider two cases:

Case 1: For every \( x \in A \), we have \( h(x) \geq b(A) \). Suppose \( A \not\subseteq \text{st}(a) \) for any \( a \in V(G) \). Then there exist \( x, y \in A \), \( h(x) \leq h(y) \), which are not related via a vertex. Let \( \sigma = (x, x_1, x_2, y) \) be a path satisfying the conclusion of Proposition 3.20, so \( \rho(x, y) = vd(x, x_1) + hpd(x_1, x_2) + vd(x_2, y) \). If \( x \) is not related via a vertex to \( x_1 \) or \( x_2 \) is not related via a vertex to \( y \) then the corresponding vertical distance in the sum is greater than or equal to \( 2^{b(x) - 1} \geq 2^{b(A) - 1} \), a contradiction. So we may assume that \( x \) is related to \( x_1 \) via vertex \( c \) and \( y \) is related to \( x_2 \) via vertex \( d \). Let \( x'_2 \in G \) such that \( x \leq_p x'_2 \) and \( h(x'_2) = h(y) \). Then by Lemma 3.17, \( vd(x, x_1) + hpd(x_2, x_2) > vd(x'_2, x_2) \). So \( \rho(x, y) > vd(x'_2, x_2) + vd(x_2, y) \geq vd(x'_2, y) \) by the triangle inequality for the partial metric \( vd \). Since \( x \) and \( y \) are not related via a vertex \( x'_2 \) and \( y \) are not related via a vertex. Thus \( \rho(x, y) > vd(x'_2, y) = 2^{b(y) - 1} \geq 2^{b(A) - 1} \), a contradiction.

Case 2: There is a point \( x \in A \) such that \( h(x) < b(A) \). Let us choose \( a \in A \) such that \( b(A) < h(a) \) and let \( v \in V(G) \) be the only point satisfying \( h(v) = b(A) \) and \( v \leq_p a \). Let \( \sigma = (x, x_1, x_2, a) \) be
a path from \( x \) to \( a \) satisfying conclusion of Proposition 3.20. First observe that \( x_2 = a \), otherwise \( \text{len}(\sigma) \geq vd(x_2, a) > 2h(A) - 1 \). Therefore \( \text{len}(\sigma) = vd(x, x_1) + hpd(x_1, a) \). Let \( v_1 \in G \) be a point such that \( h(v_1) = b(A) \) and \( x_1 \leq_p v_1 \leq_p a \). Since \( h(v_1) = h(v) = b(A) \) and \( v_1 \leq_p a \), we have \( v_1 = v \) and \( \text{len}(\sigma) = \rho(x, v) + \rho(v, a) \). If \( x \notin \text{st}(v) \), let \( v_2 \neq v \) be the point satisfying \( h(v_2) = b(A) \) and \( x \leq_p v_2 \). Then by Lemma 3.22 applied to \( v_1 \) in place of \( a, v_2 \) in place of \( b, \) and \( x \) in place of \( c \), we have \( \text{len}(\sigma) \geq \rho(x, v) > vd(v, v_2) = 2h(A) - 1 \), a contradiction. \( \square \)

**Observation 3.29.** Note that if \( \text{diam}_p(A) = 2h(A) - 1 \) then either there is \( v \in V(G) \) such that \( A \subseteq \text{st}(v) \) or for all \( x \in A, h(x) = b(A) \) and hence \( A \subseteq V(G) \).

**Observation 3.30.** Suppose \( x, y, x', y' \in G, x \neq y \), and \( a \in V(G) \) such that \( h(x) = h(y), h(x') = h(y'), h(x) \in [h(a), h(a) + 1], \) and one of the following holds:

1. \( x \leq_p x' \) and \( y \leq_p y' \) or \( x' \leq_p x \) and \( y' \leq_p y \).
2. \( \text{vd}(x, y) - \text{vd}(x', y') \geq \frac{1}{4} hpd(x, x') \).

Let \( m = \frac{\|\text{vd}(x, y) - \text{vd}(x', y')\|}{hpd(x, x')} \). From the formulas for \( \text{vd} \) and \( \text{hd} \) (which equals \( hpd \) in this case) or by Observation (3.2) we have:

1. if there is a vertex \( v \) such that \( x, y, x', y' \in R(v) \), then \( m = \frac{1}{7} \);
2. if there is a vertex \( v \) such that \( x, y, x', y' \in L(v) \), then \( m = \frac{1}{5} \);
3. otherwise \( m = \frac{1}{4} \).

**Lemma 3.31.** If \( x, y \in G \) and \( a, b \in V(G) \) such that \( h(a) \leq h(x) \leq h(y) \leq h(b) = h(a) + 1, x, y \in L(b), \) and \( \rho(x, y) > 2h(a) - 1 \), then \( hpd(x, a) < \frac{3}{4} hpd(y, a) \).

**Proof.** Let \( x_1 \) be a point such that \( \rho(x, y) = hpd(x, x_1) + vd(x_1, y) \). To simplify the calculation, let us make the following denotations. \( A = 2h(a) - 1, t = vd(x_1, y), \) and \( s = hpd(x_1, x) \). Observe that we have following: \( hpd(a, b) = 4A, hpd(y, b) = 4t, hpd(x, a) = 4A - s - 4t, \) and \( hpd(y, a) = 4A - 4t \). The assumption \( \rho(x, y) > 2h(a) - 1 \) means \( s + t > A \). Let us calculate the quotient:

\[
\frac{hpd(x, a)}{hpd(y, a)} = \frac{4A - s - 4t}{4A - 4t} < \frac{4A - A + t - 4t}{4A - 4t} = \frac{3}{4}.
\]

\( \square \)
Lemma 3.32. If \( x, y \in G \) and \( a, b \in V(G) \) such that \( h(a) \leq h(x) \leq h(y) \leq h(b) = h(a) + 1 \), \( x, y \in R(a) \), and \( \rho(x, y) > 2^{h(b) - 1} \), then \( \text{hpd}(y, b) < \frac{1}{2} \text{hpd}(x, b) \).

Proof. Let \( x_1 \) be a point such that \( \rho(x, y) = \text{vd}(x, x_1) + \text{hpd}(x_1, y) \). To simplify the calculation, let us make the following denotations. \( A = 2^{h(b) - 1} \), \( t = \text{vd}(x, x_1) \), and \( s = \text{hpd}(x_1, y) \). Observe that we have following: \( \text{hpd}(a, b) = 2A \), \( \text{hpd}(a, x) = 2t \), \( \text{hpd}(y, b) = 2A - 2t - s \), and \( \text{hpd}(x, b) = 2A - 2t \). The assumption \( \rho(x, y) > 2^{h(b) - 1} \) means \( s + t > A \).

Let us calculate the quotient:

\[
\frac{\text{hpd}(y, b)}{\text{hpd}(x, b)} = \frac{2A - 2t - s}{2A - 2t} < \frac{2A - 2t - A + t}{2A - 2t} = \frac{1}{2}.
\]

\( \square \)

To finish showing lower semi-openness of the diameter mapping, we need to define some additional functions.

Definition 3.33. For a fixed \( c \in G \), we define a function \( F_c : G \times [0, 1] \to 2^G \) by \( F_c(x, t) = \{ y \} \) if \( h(x) = h(c) \), otherwise \( y \in F_c(x, t) \) if it satisfies one of the following conditions:

1. If \( h(x) < h(c) \), then \( x \leq_p y \) and \( t = \frac{\text{hpd}(y, c)}{\text{hpd}(x, c)} \);
2. If \( h(x) > h(c) \), then \( y \leq_p x \) and \( t = \frac{\text{hpd}(y, c)}{\text{hpd}(x, c)} \).

Lemma 3.34. Suppose \( x, y \in G \), \( a, b \in V(G) \), such that \( h(a) \leq h(x) \leq h(y) \leq h(b) = h(a) + 1 \), \( \rho(x, y) > 2^{h(a) - 1} \), and \( t \in [0, 1] \). Assume moreover that \( x' \in F_a(x, t) \) and \( y' \in F_a(y, t) \). Then \( \rho(x', y') < \rho(x, y) \).

Proof. Let \( \sigma = (x, x_1, x_2, y) \) be a path satisfying the conclusion of Proposition 3.20. Denote by \( \Delta x = \text{hpd}(x, x') \), \( \Delta y = \text{hpd}(y, y') \), and note that \( \Delta x = (1 - t) \cdot \text{hpd}(x, a) \), \( \Delta y = (1 - t) \cdot \text{hpd}(y, a) \), so \( \frac{\Delta x}{\Delta y} \leq 1 \).

We need to consider four cases:

Case 1: \( x = x_1 \) and \( x_2 = y \). Then \( x, x', y' \) and \( y \) all lie on a monotone arc. Thus, by Proposition 3.6, \( \rho(x, y) - \rho(x', y') = \Delta y - \Delta x \geq 0 \). But equality would imply \( \rho(x, y) = 0 \) so \( \rho(x, y) > \rho(x', y') \).

Case 2: \( x \neq x_1 \) and \( x_2 = y \). Let \( x'_1 \in G \) such that \( x'_1 \leq_p x_1 \) and \( h(x'_1) = h(x') \). If \( \text{vd}(x', x'_1) \geq \text{vd}(x, x_1) \) then there is \( d \in V(G) \) with \( h(d) = h(b) \) and \( x', x, x'_1, x_1 \in L(d) \). But then there is \( x_2 \in G \) with \( x \leq_p x_2 \) and \( h(x_2) = h(y) \) such that \( \text{hpd}(x, x_2) + \text{vd}(x_2, y) < \rho(x, y) \) contrary to the definition of \( \rho(x, y) \). So \( \text{vd}(x', x'_1) < \text{vd}(x, x_1) \). Thus
\[ \rho(x, y) - \rho(x', y') \geq \rho(x, y) - (vd(x', x_1') + hpd(x_1', y')) = (vd(x, x_1) - vd(x', x_1')) + \Delta y - \Delta x \text{ is positive.} \]

**Case 3:** \( x = x_1 \) and \( x_2 \neq y \). Then there is a vertex \( d \) such that \( h(d) = h(b) \) and \( x_2, y \in L(d) \). Let \( x_2' \in G \) be such that \( x_2 \leq_p x_2' \) and \( h(x_2') = h(y') \). Then \( \rho(x, y) - \rho(x', y') \geq hpd(x, x_2) + vd(x_2, y) - hpd(x', x_2') - vd(x_2', y') = -\Delta x + \Delta y - \frac{\Delta y}{4} = \frac{3\Delta y}{4} - \Delta x \). Here the next to last equality is obtained by (2) of Observation 3.30 and the last expression is positive by Lemma 3.31.

**Lemma 3.35.** Suppose \( x, y \in G, a, b \in V(G), x \neq y \) such that \( h(a) \leq h(x) \leq h(y) \leq h(b) = h(a) + 1 \), \( \rho(x, y) > 2^{h(b) - 1} \), and \( t \in [0, 1] \). Assume moreover that \( x' \in F_0(x, t) \) and \( y' \in F_0(y, t) \). Then \( \rho(x', y') < \rho(x, y) \).

**Proof.** Let \( \sigma = (x, x_1, x_2, y) \) be a path satisfying the conclusion of Proposition 3.20. Denote by \( \Delta x = hpd(x, x'), \Delta y = hpd(y, y'), \) and note that \( \Delta x = (1 - t) \cdot hpd(x, b), \Delta y = (1 - t) \cdot hpd(y, b) \), so \( \frac{\Delta x}{\Delta y} \geq 1. \)

We need to consider four cases:

**Case 1:** \( x = x_1 \) and \( x_2 = y \). Then \( \rho(x, y) - \rho(x', y') = \Delta y - \Delta x > 0. \)

**Case 2:** \( x \neq x_1 \) and \( x_2 = y \). Let \( x_1' \in G \) such that \( x_1 \leq_p x_1' \) and \( h(x_1') = h(x_1) \). Consider two subcases:

**Case 2a:** There is a vertex \( c \) such that \( x, y \in R(c) \). Then \( \rho(x, y) - \rho(x', y') \geq \rho(x, y) - (vd(x', x_1') + hpd(x_1', y')) = -\frac{\Delta x}{2} + \Delta x - \Delta y = \frac{\Delta x}{2} - \Delta y \). Here the next to last equality is obtained by (2) of Observation 3.30 and the last expression is positive by Lemma 3.31.

**Case 2b:** There is no vertex as in Case 2a. Then \( \rho(x, y) - \rho(x', y') \geq \rho(x, y) - (vd(x', x_1') + hpd(x_1', y')) = -\frac{\Delta x}{4} + \Delta x - \Delta y = \frac{3\Delta x}{4} - \Delta \). Here the next to last equality is obtained by (3) of Observation 3.30 and the last expression is positive by Lemma 3.31.

**Case 3:** \( x = x_1 \) and \( x_2 \neq y \). Let \( x_2' \in G \) such that \( x_2 \leq_p x_2' \) and \( h(y') = h(x_2') \). Then there is a vertex \( d \) such that \( h(d) = h(b) \) and \( x_2, y \in L(d) \). Then \( \rho(x, y) - \rho(x', y') \geq \rho(x, y) - (hpd(x', x_2') + vd(x_2', y')) = hpd(x, x_2) + vd(x_2, y) - hpd(x', x_2') + vd(x_2', y') = \Delta x - \Delta y + \frac{3\Delta y}{4} = \Delta x - \frac{3\Delta y}{4} \). Here the next to last equality is obtained by (2) of Observation 3.30 and the last expression is positive by Lemma 3.31.
Case 4: $x \neq x_1$ and $x_2 \neq y$. Let $x'_1, x'_2 \in \mathcal{G}$ such that $x_1 \leq_p x'_1$, $x_2 \leq_p x'_2$, $h(x') = h(x'_1)$, and $h(y') = h(x'_2)$. Then $\rho(x, y) - \rho(x', y') \geq \rho(y, x) - (vd(y', x_2) + hpd(x'_2, x'_1) + vd(x_1, x')) = (vd(y, x_2) - vd(y', x_2')) + \rho(x_2, x) - (hpd(x'_2, x'_1) + vd(x_1, x'))$. The quantity $(vd(y, x_2) - vd(y', x_2'))$ is positive and the rest is also positive from previous case, so $\rho(x, y) - \rho(x', y')$ is positive. □

**Lemma 3.36.** Suppose $x, y, a \in \mathcal{G}$, $x \neq y$ such that $h(x) \leq h(a) \leq h(y)$, and $t \in [0, 1)$. Assume moreover that $x' \in F_a(x, t)$ and $y' \in F_a(y, t)$. Then $\rho(x', y') < \rho(x, y)$.

**Proof.** Let $\sigma = (x, x_1, x_2, y)$ be the path as in Proposition 3.20. Let $b \in \mathcal{G}$ satisfy $x_1 \leq_p b \leq_p x_2$ and $h(a) = h(b)$, Then by Lemma 3.22 and Lemma 3.23, $\rho(x', b) < \rho(x, b)$ and $\rho(y', b) < \rho(y, b)$, so $\rho(x', y') < \rho(x', b) + \rho(b, y) < \rho(x, b) + \rho(b, y) = \rho(x, y)$. □

**Definition 3.37.** Define $B_c : 2^\mathcal{G} \times [0, 1] \to 2^\mathcal{G}$ by $B_c(A, t) = \bigcup_{x \in A} F_c(x, t)$.

Let us observe some facts about such defined functions $B_c$.

**Observation 3.38.** For every $c \in \mathcal{G}$ and for every $A \in 2^\mathcal{G}$, we have $B_c(A, 1) = A$ and $h(B_c(A, 0)) = \{h(c)\}$. Note that the functions $F_c$ and $B_c$ depends on $h(c)$ only, so if $c, d \in \mathcal{G}$ satisfy $h(c) = h(d)$, then $F_c = F_d$ and $B_c = B_d$. Finally, note that in general, neither $F_c$ nor $B_c$ have to be continuous, but they are continuous with respect to the second variable.

**Observation 3.39.** If $s, t \in [0, 1]$ and $A \in 2^\mathcal{G}$, then $B_c(B_c(A, t), s) = B_c(A, t, s)$.

**Lemma 3.40.** Suppose $A \subseteq \mathcal{G}$ with $\text{diam}_\rho(A) > 2^{h(A)-1}$, and there is $a \in V(\mathcal{G})$ such that, every $x \in A$ we have $h(a) - 1 \leq h(x) \leq h(a) + 1$. Then $\text{diam}_\rho$ is lower semi-open at $A$.

**Proof.** From the hypothesis we see that $b(A) \geq h(a) - 1$. If $b(A) = h(a) - 1$ then for all $x \in A$, $h(x) \leq h(a)$. Let $c \in V(\mathcal{G})$ such that $h(c) = h(a) - 1$ and consider $(x, y) \in M(A)$ such that $h(x) \leq h(y)$, $t < 1$ and $x' \in F_c(x, t)$, $y' \in F_c(y, t)$ by Lemma 3.34 $\rho(x', y') < \rho(x, y)$. If $b(A) = h(a)$ then let $(x, y) \in M(A)$ such that $h(x) \leq h(y)$, $t < 1$ and $x' \in F_a(x, t)$, $y' \in F_a(y, t)$. By either Lemma 3.34 or Lemma 3.36, depending on the position of $x$ and $y$ relative to $a$, we again have $\rho(x', y') < \rho(x, y)$. This implies that $\text{diam}_\rho(B_a(A \times \{t\})) < \text{diam}_\rho(A)$ for $t < 1$, thus lower semi-openess of $\text{diam}_\rho$ follows is this case. □

**Lemma 3.41.** Suppose $A \subseteq \mathcal{G}$, $q, r \in A$ satisfy $h(q) \leq h(x) \leq h(r)$ for every $x \in A$ and assume moreover that there are $a, b \in V(\mathcal{G})$
such that $h(q) \leq h(a) < h(b) = h(a) + 1 < h(r)$. Let $c \in \mathcal{G}$ satisfy $hpd(a,c) = hpd(c,b)$. Then for any $(x,y) \in M(A)$ with $h(x) \leq h(y)$, we have $h(x) \leq h(c) \leq h(y)$.

**Proof.** Suppose on the contrary that there is $(x,y) \in M(A)$ with $h(x) \leq h(y)$ and $h(q) \leq h(x) < h(y) < h(c)$ or $h(c) < h(x) \leq h(y) \leq h(r)$.

If $h(q) \leq h(x) \leq h(y) < h(c)$, then we have $hpd(q,r) \leq \rho(x,y) \leq \text{diam}_\rho(h^{-1}([h(q), h(c)]))$.

We want to show $\text{diam}_\rho(h^{-1}([h(q), h(c)])) \leq 2^{h(c)+1} - 2^{h(q)+1} + 2^{h(q)-1}$. Let $w,z \in M(h^{-1}([h(q), h(c)]))$ with $h(w) \leq h(z)$. By Proposition 3.20, $\text{diam}_\rho(h^{-1}([h(q), h(c)])) = \rho(w,z) = vd(w,w_1) + hpd(w_1, w_2) + vd(w_2, z)$. If $h(w) = h(q)$ then we know that $\rho(w,z) \leq vd(w, w_3) + hpd(w_3, z)$ where $w_3 \leq_p z$. Since $vd(w, w_3) \leq 2^{h(q)-1}$ and $hpd(w_3, z) \leq 2^{h(c)+1} - 2^{h(q)+1}$ the inequality follows.

In the case that $h(w) > h(q)$ we have by Lemma 3.22, if $q_1 \in h^{-1}(h(q))$ and $q_1 \leq_p w$ then $\rho(w,z) < \rho(q_1,z)$, and the inequality follows from the previous.

Continuing we have $2^{h(c)+1} - 2^{h(q)+1} + 2^{h(q)-1} \leq 2^{h(r)+1} - 2^{h(c)+1} + 2^{h(c)-1}$.

In the case $h(c) < h(x) \leq h(y) \leq h(r)$ a similar argument shows $hpd(q,r) \leq \rho(x,y) \leq \text{diam}_\rho(h^{-1}([h(c), h(r)])) \leq 2^{h(r)+1} - 2^{h(c)+1} + 2^{h(c)-1}$.

In any case we have $hpd(q,r) = 2^{h(r)+1} - 2^{h(q)+1} < 2^{h(r)+1} - 2^{h(c)+1} + 2^{h(c)-1}$, therefore

$$2^{h(q)+1} > 3 \cdot 2^{h(c)-1}.$$

The condition $hpd(c,q) = hpd(c,r)$ implies that $2^{h(c)} = \frac{1}{2} \left( 2^{h(r)} + 2^{h(q)} \right)$, so substituting into (#) we get $2^{h(q)+1} > \frac{3}{4} \left( 2^{h(r)} + 2^{h(q)} \right)$, which implies $5 \cdot 2^{h(q)} > 3 \cdot 2^{h(r)}$. Since $h(r) > h(q) + 1$, then we get $5 \cdot 2^{h(q)} > 6 \cdot 2^{h(q)}$, a contradiction. \hfill \Box

**Proposition 3.42.** The diameter mapping $\text{diam}_\rho$ is lower semi-open.

**Proof.** Let $A \in 2^\mathcal{G}$. If $\text{diam}_\rho(A) < 2^{h(A)-1}$, then, by Lemma 3.28 there is vertex $v \in V(\mathcal{G})$ such that $A \subseteq \text{st}(v)$ and by Lemma 3.26 $\text{diam}_\rho$ is lower semi-open at $A$.

If $\text{diam}_\rho(A) = 2^{h(A)-1}$ then by Observation 3.29 we have $h(x) = b(A)$ and hence $x \in V(\mathcal{G})$ for all $x \in A$. Then for all $t < 1$ we have $\text{diam}_\rho(B_\rho(A \times \{t\})) = t \cdot \text{diam}_\rho(A) < \text{diam}_\rho(A)$. So $\text{diam}_\rho$ is lower semi-open at $A$.

So assume $\text{diam}_\rho(A) > 2^{h(A)-1}$. If for all $x \in A$, $h(x) \geq b(A) - 1$, then, by Lemma 3.40, $\text{diam}_\rho$ is lower semi-open at $A$. So suppose there is $q \in A$ such that $h(q) < b(A) - 1$. Then there is a point $r \in A$
and vertices $a, b \in A$ such that $h(q) \leq h(a) < h(b) \leq h(r)$. Let $c$ be a point such that $hpd(a, c) = hpd(c, b)$. By Lemma 3.41 for every $(x, y) \in M(A)$ with $h(x) \leq h(y)$, we have $h(x) \leq h(c) \leq h(y)$. Therefore, by Lemma 3.36, for every $t < 1$ we have $\text{diam}_\rho(B_a(A \times \{t\})) < \text{diam}_\rho(A)$, so $\text{diam}_\rho$ is lower semi-open at $A$.

Proof of Theorem 3.1 follows immediately from Propositions 3.25 and 3.42.

4. Metrics on all graphs

As a consequence of Theorem 3.1 we get the following corollary.

**Corollary 4.1.** A graph $G$ admits a metric for which the diameter mapping is open if and only if all components of $G$ are nondegenerate or all components of $G$ are degenerate i.e. $G$ is a discrete finite graph.

**Proof.** If $G$ is a graph whose all components are nondegenerate, then by Theorem 3.1 each component $C$ of $G$ admits a metric $\rho_C$ such that the diameter mapping is open. We may additionally assume that $\text{diam}_{\rho_C}(C) = 1$. Define $\rho : G \times G \to [0, 1] \cup \{2\}$ by $\rho(x, y) = \rho_C(x, y)$ if $x, y$ belong to the same component $C$, and $\rho(x, y) = 2$ if $x$ and $y$ do not belong to the same component. One can verify that the diameter map for $\rho$ is an open mapping from $2^G$ onto $[0, 1] \cup \{2\}$.

If $G$ is a discrete graph, then the discrete metric $\rho$ gives open diameter onto the the set $\{0, 1\}$.

It remains to show that for a graph $G$ that contains a nondegenerate component $C$ and a degenerate component $\{d\}$ there is no metric that induces open diameter mapping. Let $\rho$ be any metric on $G$. Then $\text{diam}_\rho(2^C)$ is an open subset of the range of $\text{diam}_\rho$ and it is an interval of the form $[0, \text{diam}_\rho(C)]$. Similarly, because $\{d\}$ is an open subset of $2^G$, the set $\{0\}$ has to be an open subset of the range, a contradiction. □

5. Problems

In this section we gather some open problems about openness of the diameter mappings. Some of them, that remain unsolved, we repeat from [5].

As a consequence of [5, Corollary 5.18] every dendrite that contains a free arc admits an open diameter mapping. We do not know if the assumption of containing a free arc is essential.

**Problem 5.1.** Does every dendrite admit a metric for which the diameter is open? In particular, does the dendrite $D_3$ (i.e. the dendrite with a dense set of ramification points, each of order 3) admits such a metric?
More generally, we do not know any locally connected continuum that does not admit open diameter mapping.

**Problem 5.2.** Does every locally connected continuum admit a metric for which the diameter is open?

**Problem 5.3.** Do all contractible continua admit a metric for which the diameter is open?

Let us pose a general problem, however we know that we are far from the solution.

**Problem 5.4.** Characterize continua that admit a metric for which the diameter is open.

**References**


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