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Openness of induced mappings

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Abstract

Openness of induced mappings between hyperspaces of continua is studied. In particular we investigate continua X such that if for a mapping $f: X \to Y$ the induced mapping $C(f): C(X) \to C(Y)$ is open, then f is a homeomorphism. It is shown that, besides hereditarily locally connected continua, all fans have this property, while some Cartesian products do not have it. If $f: X \times Y \to X$ denotes the natural projection, then openness of $C(f)$ implies that X is hereditarily unicoherent. The equivalence holds for dendrites. Some new characterizations of these curves are obtained.

Keywords: Continuum; Dendrite; Hyperspace; Induced mapping; Open

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All spaces considered in this paper are assumed to be metric. A *mapping* means a continuous function. To exclude some trivial statements we assume that all considered mappings are not constant. A *continuum* means a compact connected space. Given a continuum X with a metric d, we let 2^X to denote the hyperspace of all nonempty closed subsets of X equipped with the Hausdorff metric H defined by

$$H(A,B) = \max\big\{\sup\big\{d(a,B):\ a \in A\big\},\ \sup\big\{d(b,A):\ b \in B\big\}\big\}$$

(see, e.g., [22, (0.1), p. 1 and (0.12), p. 10]). Further, we denote by $C(X)$ the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X, and the symbol $C^2(X)$ stands for $C(C(X))$. The reader is referred to Nadler's book [22] for needed information on the structure of hyperspaces.

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Given a mapping $f : X \to Y$ between continua X and Y, we consider mappings (called the *induced* ones)

$$2^f : 2^X \to 2^Y \quad \text{and} \quad C(f) : C(X) \to C(Y)$$

defined by

$$2^f(A) = f(A) \qquad \text{for every } A \in 2^X \quad \text{and}$$

$$C(f)(A) = f(A) \quad \text{for every } A \in C(X),$$

and the induced mapping $C^2(f) : C^2(X) \to C^2(Y)$ is defined correspondingly.

A mapping $f : X \to Y$ between spaces X and Y is said to be *open* provided the image of an open subset of the domain is open in the range. The following results concerning induced mappings for the class of open mappings are known (see [11, Theorem 4.3]; compare also [10, Theorem 3.2]).

Statement 1. *Let a surjective mapping $f : X \to Y$ between continua X and Y be given. Consider the following conditions*:

(a) *$f : X \to Y$ is open*;
(b) *$C(f) : C(X) \to C(Y)$ is open*;
(c) *$2^f : 2^X \to 2^Y$ is open.*

Then (a) *and* (c) *are equivalent, and each of them is implied by* (b).

An example is known [11, Section 4] of open surjective mappings $f : X \to Y$ between locally connected continua X and Y such that the induced mapping $C(f) : C(X) \to C(Y)$ is not open. It is so because of the following result [7, Theorem 1], in which a *monotone* mapping means one with connected point-inverses.

Statement 2. *If a continuum X is locally connected, and for a mapping $f : X \to Y$ the induced mapping $C(f) : C(X) \to C(Y)$ is open, then f is monotone.*

As a consequence of this statement the following corollary has been shown in [7, Corollary 2].

Corollary 3. *Let a continuum X be hereditarily locally connected, and a mapping $f : X \to Y$ be such that the induced mapping $C(f) : C(X) \to C(Y)$ is open. Then f is a homeomorphism.*

Therefore the following question seems to be of some interest.

Question 4. What (locally connected) continua X have the property that if f is a mapping of X onto a continuum Y such that the induced mapping $C(f) : C(X) \to C(Y)$ is open, then f is a homeomorphism?

As a more particular question one may ask the following.

Question 5. For what locally connected continua X every open and monotone mapping on X is a homeomorphism?

If openness of $C^2(f)$ is assumed in place of that of $C(f)$, then the conclusion of Corollary 3 holds with no assumption on the continuum X (see [13, Theorem]).

Statement 6. *For an arbitrary continuum X and a mapping $f:X\to Y$ if the induced mapping $C^2(f):C^2(X)\to C^2(Y)$ is open, then f is a homeomorphism.*

Thus a natural problem arises to find possible consequences of openness of the induced mapping $C(f):C(X)\to C(Y)$ under some additional assumptions concerning either continua X and/or Y, or the mapping f itself. The aim of the paper is to present further results in this direction.

Given a (metric) space X we denote by d_X the metric on X, and by $B_X(p,\varepsilon)$ the (open) ball in X centered at a point $p\in X$ and having the radius ε. Given a subset $A\subset X$, we define

$$N_X(A,\varepsilon)=\bigcup\bigl\{B_X(a,\varepsilon):\ a\in A\bigr\},$$

and we use the symbols $\mathrm{cl}_X(A)$, $\mathrm{int}_X A$, $\mathrm{bd}_X A$ and $\mathrm{diam}_X A$ to denote the closure, the interior, the boundary and the diameter of A in X, respectively. The symbol $\mathbb{N}$ stands for the set of all positive integers.

To show a class of non-locally connected continua X for which openness of the induced mapping $C(f):C(X)\to C(Y)$ forces $f:X\to Y$ to be a homeomorphism (see Corollary 3 above) some definitions are in order first. A continuum the intersection of every two subcontinua of which is connected is said to be *hereditarily unicoherent*. A continuum is called a *dendroid* provided that it is hereditarily unicoherent and arcwise connected. Given points a and b in a dendroid X, we denote by ab the (unique) arc in X joining these points. An *end point* of a dendroid X is defined as a point p of X which is an end point of each arc containing p. By a *ramification point* of a dendroid X we understand a point which is the center of a simple triod contained in X. A dendroid having exactly one ramification point v is called a *fan*, and v is called its *top*. A mapping $f:X\to Y$ between continua X and Y is said to be *monotone relative to a point* $p\in X$ provided that for each subcontinuum Q of Y such that $f(p)\in Q$ the inverse image $f^{-1}(Q)$ is connected. It is known that if X and Y are dendroids, then $f:X\to Y$ is monotone relative to $p\in X$ if and only if the restriction $f|px$ is monotone for each point $x\in X$ (see [17, Corollary 2.10, p. 722]).

Lemma 7. *Let X be a fan with the top v, and let a mapping $f:X\to Y$ be monotone relative to v and such that the induced mapping $C(f):C(X)\to C(Y)$ is open. Then f is a homeomorphism.*

Proof. Since (b) implies (a) in Statement 1, it follows that f is open. Thus by Proposition 3.4 of [6, p. 12] we infer that the range space Y is either a fan or an arc. So, by [17, Corollary 2.10, p. 722], the restriction $f|vx$ is monotone for each point $x\in X$.

Since each monotone open mapping between fans is a homeomorphism, and since there is no monotone open mapping of a fan onto an arc (see [6, Theorem 7.11, p. 39]) we may assume that f is not monotone. Therefore, there is a point $y \in Y$ whose inverse image $f^{-1}(y)$ is not connected. Choose two points $x_1, x_2 \in X$ in two distinct components of $f^{-1}(y)$. Thus $x_1 \notin vx_2$ and $x_2 \notin vx_1$. Put $K = vx_1 \cup vx_2$. Let ε be a positive number satisfying $d_Y(f(v), f(x_1)) > 2\varepsilon$. By local connectedness of $f(K)$ there is an $\varepsilon' > 0$ such that if $d_Y(y, f(x_1)) < \varepsilon'$ for $y \in f(K)$, then the arc $yf(x_1)$ is contained in the ball $B_Y(f(x_1), \varepsilon)$. Let $\delta > 0$ satisfy the definition of continuity of f for this ε'. Take $\eta > 0$ such that $\eta < \delta$ and that the conditions $d_X(x_1, x_1') < \eta$ and $d_X(x_2, x_2') < \eta$ imply that the arc $x_1'x_2'$ contains a point v' with $d_X(v, v') < \delta$. By openness of $C(f)$ the image $C(f)(B_{C(X)}(K, \eta))$ is open. Then there exists $L \in C(f)(B_{C(X)}(K, \eta))$ such that $f(v) \notin L$. Let $K' \in B_{C(X)}(K, \eta)$ be such that $f(K') = L$. Since $v \notin K'$, the continuum K' is an arc (or a point). Let e be an end point of X such that $K' \subset ve$. Then by the assumption the restriction $f|ve$ is monotone, hence hereditarily monotone [18, (6.10), p. 53].

Take points $x_1', x_2' \in K'$ that satisfy $d_X(x_1, x_1') < \eta$ and $d_X(x_2, x_2') < \eta$. Then the arc $x_1'x_2'$ contains a point v' such that $d_X(v, v') < \delta$. Hence

$$d_Y\big(f(x_1), f(x_1')\big) < \varepsilon', \quad d_Y\big(f(v), f(v')\big) < \varepsilon', \quad d_Y\big(f(x_2), f(x_2')\big) < \varepsilon'.$$

Then

$$f(x_1')f(x_2') \subset B_Y\big(f(x_1'), \varepsilon\big) \quad \text{and} \quad f(v') \notin B_Y\big(f(x_1'), \varepsilon\big).$$

On the other hand, $f|x_1'x_2'$ is monotone, so $f(v') \in f(x_1')f(x_2') \subset B_Y(f(x_1'), \varepsilon)$. This contradiction completes the proof. □

The following known result (see [6, Theorem 5.7, p. 21]) will be used in the next proof. We rewrite it here for the reader's convenience. Recall that a mapping $f: X \to Y$ between continua X and Y is said to be *confluent* provided that for each subcontinuum Q of Y each component of $f^{-1}(Q)$ is mapped onto Q under f; monotone, as well as open mappings are known to be confluent [6, Fact 3.0, p. 11].

Statement 8. *Let a surjective confluent mapping $f: X \to Y$ be defined on a fan X with the top v. If Y is an arc, then for each end point e of X either $f|ve$ is constant, or there is a finite sequence of distinct points $v = x_0, x_1, \ldots, x_n = e$ in ve ordered from v to e such that for each $k \in \{0, 1, \ldots, n-1\}$ the mapping $f|x_kx_{k+1}: x_kx_{k+1} \to Y$ is monotone; if $k \neq 0$, then $f|x_kx_{k+1}$ is a surjection onto the whole arc Y, and moreover, if $f(v)$ is an end point of Y, then also $f|x_0x_1$ is a surjection.*

Theorem 9. *Let a continuum X be a fan, and let $f: X \to Y$ be such a mapping that the induced mapping $C(f): C(X) \to C(Y)$ is open. Then f is a homeomorphism.*

Proof. Let v be the top of the fan X. By Lemma 7 we may assume that f is not monotone relative to v. Since f is open by the implication from (b) to (a) of Statement 1, and since each confluent (thus each open) mapping $f: X \to Y$ onto a fan Y is monotone relative to v

(see [5, Lemma 4, p. 32]; also [6, Theorem 4.1, (5), p. 14]), we infer that Y is not a fan. Thus by Proposition 3.4 of [6, p. 12] Y is an arc. Put $Y = ab$. We may assume that $f(v) \neq a$. Since f is not monotone relative to v, hence by Statement 8 there are two points $x_1, x_2 \in X$ such that $x_1 \in vx_2$, $f(x_1) = a$, $f(x_2) = b$, and the restrictions $f|vx_1 : vx_1 \to f(v)a$ and $f|x_1x_2 : x_1x_2 \to ab$ are monotone. Let $t \in x_1x_2$ be such that $f(t) = f(v)$, and put $K = vt$. Let $\varepsilon > 0$ be such that $d_Y(a, f(v)) > 2\varepsilon$. By local connectedness of K there exists an $\varepsilon' > 0$ with $\varepsilon' < \varepsilon$, and such that for each $y \in f(K)$ if $d_Y(y, f(v)) < \varepsilon'$, then the arc $yf(v)$ is contained in the ball $B_Y(f(v), \varepsilon)$. Let $\delta > 0$ satisfy the definition of continuity of f for this ε'. Take $\eta > 0$ such that $\eta < \delta$ and that the conditions $d_X(v, v') < \eta$ and $d_X(t, t') < \eta$ imply that the arc $v't' \subset X$ contains a point x_1' with $d_X(x_1, x_1') < \delta$. By openness of $C(f)$ the image $C(f)(B_{C(X)}(K, \eta))$ is open. Then there exists $L \in C(f)(B_{C(X)}(K, \eta))$ such that $a, f(v) \notin L$. Let $K' \in B_{C(X)}(K, \eta)$ be such that $f(K') = L$. Since $v \notin K'$, the continuum K' is an arc (or a point). Take v' and t' in K' that satisfy $d_X(v, v') < \eta$ and $d_X(t, t') < \eta$. Then the arc $v't'$ contains a point x_1' such that $d_X(x_1, x_1') < \delta$. Therefore

$$d_Y\big(f(v), f(v')\big) < \varepsilon', \quad d_Y\big(a, f(x_1')\big) < \varepsilon', \quad d_Y\big(f(t), f(t')\big) < \varepsilon'.$$

Then $f(v')f(t')$ is contained in the ball $B_Y(f(v), \varepsilon)$ and $f(x_1') \notin B_Y(f(v), \varepsilon)$. On the other hand the restriction $f|v't'$ is monotone, so $f(x_1') \in f(v')f(t') \subset B_Y(f(v), \varepsilon)$. This contradiction finishes the proof. □

Recall that a continuum is called a *λ-dendroid* if it is hereditarily unicoherent and hereditarily decomposable. Each dendroid is known to be a λ-dendroid, and the both concepts are preserved under confluent (so under open) mappings (see, e.g., [18, Table IV, p. 69; (7.30), p. 66 and (7.24), p. 64]).

Question 10. Let a continuum X be (a) a dendroid, or (b) a λ-dendroid, and let a mapping $f : X \to Y$ be such that $C(f) : C(X) \to C(Y)$ is open. Does then it follow that f is a homeomorphism?

Note that the answer to Question 10 is affirmative if X is additionally assumed to be either a *dendrite* (i.e., a locally connected dendroid, which is known to be hereditarily locally connected) or a fan—see Corollary 3 and Theorem 9 above.

To formulate the next result we need two more definitions. A mapping $f : X \to Y$ between continua X and Y is said to be:

- *interior at a point* $p \in X$ provided that for each open set U containing p the image $f(p)$ is an interior point of $f(U)$ (note that f is open if and only if it is interior at each point $p \in X$);
- *atomic* provided that for each subcontinuum K of X either $f(K)$ is a singleton or $f^{-1}(f(K)) = K$ (it is known [18, (4.14), p. 17] that every atomic mapping of a continuum is monotone).

We describe more examples of open mappings f between continua X and Y such that the induced mapping $C(f)$ is not open, but with another reason of non-openness of f. Namely in case of locally connected continua or fans the mapping $C(f)$ was not interior at some nondegenerate subcontinua of X, while now we exhibit a mapping f such that $C(f)$ is not interior at singletons.

Theorem 11. *For an arbitrary continuum X if a mapping $f:X\to Y$ is atomic and such that for each point $x\in X$ the induced mapping $C(f):C(X)\to C(Y)$ is interior at $\{x\}$, then f is a homeomorphism.*

Proof. Suppose the contrary and take a point $y\in Y$ such that the continuum $f^{-1}(y)$ is nondegenerate. Let $x\in f^{-1}(y)$ and let $\varepsilon>0$ be such that $2\varepsilon<\operatorname{diam}_X f^{-1}(y)$. Since $C(f)$ is interior at $\{x\}$, we have $\{y\}\in \operatorname{int}_{C(Y)} C(f)(B_{C(X)}(\{x\},\varepsilon))$. Thus there is a nondegenerate continuum $L\in C(f)(B_{C(X)}(\{x\},\varepsilon))$ such that $y\in L$. Let $K\in B_{C(X)}(\{x\},\varepsilon)$ be such that $f(K)=L$. By atomicity of f we have $K=f^{-1}(f(K))\supset f^{-1}(y)$, whence $\operatorname{diam}_X K>2\varepsilon$, contrary to the condition $H(K,\{x\})<\varepsilon$. The proof is then finished. □

As an example of the situation described in Theorem 11 one can consider a curve of pseudo-arcs as constructed in [16, p. 93] (compare also the arc of pseudo-arcs in [3, p. 173], where this continuum was named “a continuous snake-like arc of pseudo-arcs”). More precisely, we have the following corollary.

Corollary 12. *Let Y be a curve, and let X be the curve Y of pseudo-arcs* (*in particular, X can be an arc of pseudo-arcs*). *If $f:X\to Y$ denotes the natural projection, then the induced mapping $C(f)$ is not open.*

Proof. Really, the projection is both open and atomic, but not a homeomorphism, so the conclusion follows from Theorem 11. □

The previous results, viz. Corollary 3, Theorems 9 and 11, and Corollary 12, as well as Example 3.2 of [9, p. 4] indicate that there are various types of continua, simple ones and having more complicated structure, admitting an open mapping f onto a continuum such that the induced mapping $C(f)$ is not open. So, one can ask if this phenomenon occurs always when the considered open mapping f is not a homeomorphism. Our next result shows that this is not the case: there are open mappings f of (locally connected) continua, having nondegenerate point inverses, and such that the induced mapping $C(f)$ is also open.

Proposition 13. *For arbitrary continua X and Y let $f:X\times Y\to X$ denote the natural projection. Then the induced mapping $C(f):C(X\times Y)\to C(X)$ is interior at every $K\in C(X\times Y)$ such that $f(K)$ is a singleton.*

Proof. The mapping f is defined by $f((x,y))=x$. If $f(K)$ is a singleton, then $K=\{x_0\}\times Y_0$ for some $x_0\in X$ and some continuum $Y_0\subset Y$. If a continuum $Q\subset X$ satisfies $H(\{x_0\},Q)<\varepsilon$, then putting $L=Q\times Y_0$ we have $H(K,L)<\varepsilon$ and $f(K)=Q$. This means that the image of $B_{C(X\times Y)}(K,\varepsilon)$ under $C(f)$ contains the ball $B_{C(X)}(\{x_0\},\varepsilon)$, so $C(f)$ is interior at K. The argument is complete. □

Recall that if P and Q are subspaces of a metric space X with a metric d, and ε is a positive number, then a mapping $g : P \to Q$ is called an *ε-translation* provided that $d(p, g(p)) < \varepsilon$ for each point $p \in P$.

Proposition 14. *Let a continuum X and its nondegenerate subcontinuum P satisfy the following condition.*

(15) *For each $\varepsilon > 0$ there is a $\delta > 0$ such that for each subcontinuum Q of X satisfying $H(P, Q) < \delta$ there exists a surjective ε-translation $g : P \to Q$.*

For an arbitrary continuum Y let $f : X \times Y \to X$ denote the natural projection. Then the induced mapping $C(f) : C(X \times Y) \to C(X)$ is interior at each continuum K for which $f(K) = P$.

Proof. Take any $\varepsilon > 0$, and let $\delta > 0$ be as in condition (15). Take an ε-ball $\mathcal{B} = B_{C(X\times Y)}(K, \varepsilon)$. It is enough to prove that $C(f)(\mathcal{B})$ contains a δ-neighborhood of the point $C(f)(K) = P$ in the range hyperspace $C(X)$. So let a subcontinuum Q of X be a point of the ball $B_{C(X)}(P, \delta)$. This means that $H(P, Q) < \delta$. Therefore by (15) there exists a surjective ε-translation $g : P \to Q$. Define $L = \{(g(x), y) \in Q \times Y\colon (x, y) \in K\}$, and note that L is a continuum with $H(K, L) < \varepsilon$ and $f(L) = Q$. The proof is finished. □

Propositions 13 and 14 imply the following theorem.

Theorem 16. *Let a continuum X satisfy the following condition.*

(17) *For each nondegenerate subcontinuum P of X and for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each subcontinuum Q of X satisfying $H(P, Q) < \delta$ there exists a surjective ε-translation $g : P \to Q$.*

Then X has the following property.

(18) *For each continuum Y, if $f : X \times Y \to X$ denotes the natural projection, then the induced mapping $C(f) : C(X \times Y) \to C(X)$ is open.*

Note that every arc (in particular the closed unit interval [0, 1]) satisfies condition (17). Thus we have the following corollary.

Corollary 19. *Let Y be a continuum and let $f : [0, 1] \times Y \to [0, 1]$ denote the natural projection. Then the induced mapping $C(f) : C([0, 1] \times Y) \to C([0, 1])$ is open.*

There are continua X which do not have property (17). Such is, for example, the square $[0, 1] \times [0, 1]$, because if P and Q are its subcontinua such that P is locally connected while Q is not, then there is no mapping from P onto Q. The next results give more information about it.

Theorem 20. *Let X and Y be nondegenerate continua, P be a subcontinuum of X, and $f : X \times Y \to X$ denote the natural projection. If $C(f)$ is interior at every subcontinuum K of $C(X \times Y)$ for which $f(K) = P$, then P is unicoherent.*

Proof. Assume the contrary. Let Q and R be two subcontinua of P such that $P = Q \cup R$ and $Q \cap R$ is not connected, i.e., $Q \cap R = A \cup B$, where A and B are nonempty disjoint closed subsets of P. Using Kuratowski–Zorn Lemma we can choose R in such a way that it is minimal in the sense that no proper subcontinuum R' of Q intersects both A and B.

Choose $a \in A$ and $b \in B$, and let y_1 and y_2 be two distinct points of Y. Define

$$K = \bigl(Q \times \{y_1\}\bigr) \cup \bigl(\{a\} \times Y\bigr) \cup \bigl(R \times \{y_2\}\bigr).$$

We will show that $C(f)$ is not interior at K.

Let U and V be open subsets of X such that $A \subset U$, $B \subset V$ and $\mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) = \emptyset$. Then $(Q \setminus (U \cup V)) \cap R = \emptyset = (R \setminus (U \cup V)) \cap Q$. Thus there exist two open subsets S and T of X such that

$$\begin{aligned}
&Q \setminus (U \cup V) \subset S \subset \mathrm{cl}_X(S) \subset X \setminus R,\\
&R \setminus (U \cup V) \subset T \subset \mathrm{cl}_X(T) \subset X \setminus Q,\\
&\mathrm{cl}_X(S) \cap \mathrm{cl}_X(T) = \emptyset.
\end{aligned}$$

Let W_1 and W_2 be open subsets of Y such that $y_1 \in W_1$ and $y_2 \in W_2$ and $\mathrm{cl}_Y(W_1) \cap \mathrm{cl}_Y(W_2) = \emptyset$.

Let

$$M = \bigl[(S \cup U \cup V) \times W_1\bigr] \cup [U \times Y] \cup \bigl[(T \cup U \cup V) \times W_2\bigr],$$

and let

$$\mathcal{M} = \bigl\{D \in C(X \times Y):\ D \subset M\bigr\}.$$

Then $\mathcal{M}$ is an open subset of $C(X \times Y)$, and $K \in \mathcal{M}$.

In order to prove that $C(f)$ is not interior at K it is enough to show that

$$P = Q \cup R = C(f)(K) \notin \mathrm{int}_{C(X)}\, C(f)(\mathcal{M}).$$

Suppose the contrary. Then there exists a proper subcontinuum R_0 of R such that $b \in R_0$ and $Q \cup R_0 \in \mathrm{int}_{C(X)}\, C(f)(\mathcal{M})$. We may assume also that $R_0 \cap U \neq \emptyset$. By the minimality of R we see that $R_0 \cap A = \emptyset$.

Let $K_0 \in \mathcal{M}$ be such that $f(K_0) = Q \cup R_0$. Define

$$\begin{aligned}
&K_1 = K_0 \cap \bigl(\bigl[\mathrm{cl}_X(V \cup Q) \times \mathrm{cl}_Y(W_1)\bigr] \cup \bigl[(Q \cap \mathrm{cl}_X(U)) \times Y\bigr]\bigr) \quad \text{and}\\
&K_2 = K_0 \cap \bigl(\bigl[\mathrm{cl}_X(V \cup R_0) \times \mathrm{cl}_Y(W_2)\bigr] \cup \bigl[(R_0 \cap \mathrm{cl}_X(U)) \times Y\bigr]\bigr).
\end{aligned}$$

We will show that $K_0 = K_1 \cup K_2$ is a separation of K_0 which will contradict the connectedness of K_0. Clearly, K_1 and K_2 are closed.

Take $p = (x, y) \in K_0$. Then $x \in Q \cup R_0$. If $x \in \mathrm{cl}_X(U)$, then clearly $p \in K_1 \cup K_2$. Suppose then that $x \notin U$. Since $p \in M$, we may assume that $p \in (S \cup U \cup V) \times W_1$ (the case when $p \in (T \cup U \cup V) \times W_2$ is similar). If $x \in V$, then clearly $p \in K_1$. So we may assume that $x \in S \subset X \setminus R$. Then $x \in Q$. Thus $p \in K_1$. We have proved that $K_0 = K_1 \cup K_2$.

Since $\mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) = \emptyset$ and $Q \cap R_0 \cap \mathrm{cl}_X(U) \subset A \cap R_0 = \emptyset$, it follows that $K_1 \cap K_2 = \emptyset$. Since $Q \cap U \neq \emptyset \neq R_0 \cap U$, we see that $K_1 \neq \emptyset \neq K_2$. Therefore we have obtained a separation of K_0. This contradiction concludes the proof. □

The next result is a consequence of Theorem 20.

Theorem 21. *Let X and Y be nondegenerate continua, and let $f : X \times Y \to X$ denote the natural projection. If $C(f)$ is open, then X is hereditarily unicoherent.*

The converse implication to that of Theorem 21 is not true. The next example shows this.

Example 22. There exists a hereditarily unicoherent continuum X such that for each nondegenerate continuum Y if $f : X \times Y \to X$ denotes the natural projection, then $C(f)$ is not open.

Proof. Given two points p and q in the plane or in the three space, we denote by pq the straight line segment having p and q as its end points. Put $v = (0,0)$, $a = (-1,0)$, $b = (0,1)$, $c = (1,0)$, and let $T = va \cup vb \cup vc$. Then T is a simple triod. For each $n \in \mathbb{N}$ put $p_n = (-1/n, 1/n)$, $b_n = (0, 1+1/n)$, $q_n = (1/n, 1/n)$, $c_n = (1, 1/n)$. Thus the unions $L_n = ap_n \cup p_nb_n \cup b_nq_n \cup q_nc_n$ are broken lines in the upper half-plane that approximate the triod T. Define $X = T \cup \bigcup\{L_n\colon n \in \mathbb{N}\}$. Hence X is a dendroid.

Let y_1 and y_2 be two distinct points of Y. In the product $X \times Y$ we distinguish a continuum K defined by $K = (ac \times \{y_2\}) \cup (\{c\} \times Y) \cup (vc \times \{y_1\}) \cup (vb \times \{y_1\})$. We will show that $C(f)$ is not interior at K. Indeed, observe that $T = f(K)$, the continua L_n tend to T, and (for sufficiently great n) they are not projections of continua close to K. Really, for sufficiently small $\varepsilon > 0$ and sufficiently great $n \in \mathbb{N}$ the sets $(L_n \times Y) \cap B_{X\times Y}((a, y_2), \varepsilon)$ and $(L_n \times Y) \cap B_{X\times Y}((c, y_2), \varepsilon)$ are in different components of $(L_n \times Y) \cap N_{X\times Y}(K, \varepsilon)$. □

In the light of Theorems 16 and 21 the following two questions are natural.

Question 23. What continua X have property (17)?

Question 24. What continua X have property (18)?

Our next results give some partial answers to these questions. To formulate and prove these results we recall some auxiliary concepts. If A is a subset of a hereditarily unicoherent continuum X, let $I(A)$ denote the minimal continuum containing A, i.e., the intersection of all subcontinua of X that contain A. It is well known that for hereditarily unicoherent continua $I(A)$ is uniquely determined [4, T1, p. 187].

A metric space X equipped with a metric d is said to be *convex* (and then d is called a *convex metric* on X) if for every two distinct points x and y of X there exists a point $z \in X$ different from x and y and such that $d(x, y) = d(x, z) + d(z, y)$. It is well known that each locally connected continuum admits a convex metric (that is equivalent to the original one; see [1, Theorem 8, p. 1109], [2, Theorem 6, p. 546], [20, Theorem 4, p. 1119], and [21]). Thus, in particular, *each dendrite admits a convex metric*. This fact can also be deduced from an earlier result in [15, p. 324].

The following lemma is a consequence of the definitions.

Lemma 25. *Let A and B be closed subsets of a dendrite X, let H stands for the Hausdorff metric induced by a convex metric on X, and let $\varepsilon > 0$ be given. If $H(A, B) < \varepsilon$, then $H(A, I(A \cup B)) < \varepsilon$.*

Lemma 26. *Let A and B be dendrites with $A \subset B$, and let H stands for the Hausdorff metric induced by a convex metric d on B. For each $\varepsilon > 0$ if $H(A, B) < \varepsilon$, then there exists a 3ε-translation $f : A \to B$ of A onto B.*

Proof. Let $\varepsilon > 0$ be given. By the Sierpiński characterization of locally connected continua (see, e.g., [14, §50, II, Theorem 2, p. 256]) the dendrite A is the union of finitely many, say n, nondegenerate subdendrites A_k with $\operatorname{diam}_B A_k < \varepsilon$ for each $k \in \{1, \ldots, n\}$. We may assume $\operatorname{card}(A_i \cap A_j) \leqslant 1$ for every two distinct indices i and j. For each point $b \in B$ let $L(b, A)$ stand for the unique arc that joins b with A, i.e., such that $A \cap L(b, A)$ is a singleton being the other end point of the arc (if $b \in A$, then $L(b, A) = \{b\}$ by the definition). For each index $k \in \{1, \ldots, n\}$ put $B_k = \{b \in B\colon\ A_k \cap L(b, A) \neq \emptyset\}$. Then B_k are dendrites, and

$$A_k \subset B_k, \qquad \operatorname{diam}_B B_k \leqslant 3\varepsilon, \qquad B = \bigcup \bigl\{B_k\colon\ k \in \{1, \ldots, n\}\bigr\}.$$

Since for each $k \in \{1, \ldots, n\}$ the sets A_k and B_k are absolute retracts [14, §53, III, Theorem 16, p. 344], hence absolute extensors [12, Chapter 3, Theorems 3.1 and 3.2, pp. 83 and 84], there exists a surjective mapping $f_k : A_k \to B_k$ such that the restriction $f_k| \operatorname{bd}_A A_k$ is the identity mapping. Note that since $\operatorname{diam}_B B_k \leqslant 3\varepsilon$, the mapping f_k is a 3ε-translation. Define $f : A \to B$ as the union of the mappings f_k, for $k \in \{1, \ldots, n\}$. Then f is a 3ε-translation of A onto B. □

Theorem 27. *Each dendrite has property* (17).

Proof. Let X be a dendrite equipped with a convex metric d. For each subdendrite P of X and each $\varepsilon > 0$ we take $\delta = \varepsilon/4$. Let P and Q be subdendrites of X such that $H(P, Q) < \delta$, where H is the Hausdorff metric induced by the (convex) metric d on X. Put $C = I(P \cup Q)$. Then $H(P, C) < \delta$ by Lemma 25. By Lemma 26 there exists a surjective 3δ-translation $f : P \to C$. Let $g : C \to Q$ be the monotone retraction [8, Theorem, p. 157]. Then by Lemma 25 we have $H(Q, C) < \delta$, and by the convexity of the metric d on X, the mapping g is a δ-translation. Then the composition $g \circ f : P \to Q$ is a $4\delta = \varepsilon$-translation, and so X has property (17). The proof is finished. □

Theorem 28. *Consider the following conditions that a continuum X and its nondegenerate subcontinuum P may satisfy*:

(15) *For each $\varepsilon > 0$ there is a $\delta > 0$ such that for each subcontinuum Q of X satisfying $H(P, Q) < \delta$ there exists a surjective ε-translation $g : P \to Q$.*

(29) *For each nondegenerate continuum Y if $f : X \times Y \to X$ denotes the natural projection, the induced mapping $C(f) : C(X \times Y) \to C(X)$ is interior at every continuum K satisfying $f(K) = P$.*

(30) *There exists a nondegenerate continuum Y such that if $f : X \times Y \to X$ denotes the natural projection, then the induced mapping $C(f) : C(X \times Y) \to C(X)$ is interior at every continuum K satisfying $f(K) = P$.*

(31) *P is unicoherent.*

Then the following implications hold:

$$(15) \Rightarrow (29) \Rightarrow (30) \Rightarrow (31).$$

Proof. The implication from (15) to (29) is shown in Propositions 13 and 14. The one from (29) to (30) is trivial. And (30) implies (31) by Theorem 20. □

The next result is a consequence of the previous one.

Theorem 32. *Consider the following conditions that a continuum X may satisfy*:

(17) *For each nondegenerate subcontinuum P of X and for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each subcontinuum Q of X satisfying $H(P, Q) < \delta$ there exists a surjective ε-translation $g : P \to Q$.*

(18) *For each continuum Y if $f : X \times Y \to X$ denotes the natural projection, the induced mapping $C(f) : C(X \times Y) \to C(X)$ is open.*

(33) *There exists a nondegenerate continuum Y such that if $f : X \times Y \to X$ denotes the natural projection, then the induced mapping $C(f) : C(X \times Y) \to C(X)$ is open.*

(34) *X is hereditarily unicoherent.*

Then the following implications hold:

$$(17) \Rightarrow (18) \Rightarrow (33) \Rightarrow (34).$$

Moreover, if the continuum X is locally connected, then (34) *implies* (17), *and therefore all four conditions are equivalent.*

Proof. The implication from (17) to (18) is shown in Theorem 16. The one from (18) to (33) is trivial. And (33) implies (34) by Theorem 21.

If X is locally connected, then assuming (34) we see that X is a dendrite [19, Chapter X, §2, Theorems 1 and 2, p. 306], whence (17) follows by Theorem 27. □

Corollary 35. *For locally continua X every one of conditions* (17), (18) *and* (33) *is equivalent to X be a dendrite.*

The authors do not know whether the implication from (18) to (33) in Theorem 32 can be reversed. So, the following question is open.

Question 36. Are conditions (18) and (33) equivalent?

Further, Example 22 shows that (34) does not imply (33), and (18) does not imply (17) by the example below. Recall that $\mathbb{R}$ stands for the real line, and $\|\cdot\|$ denotes the Euclidean norm in the plane $\mathbb{R}^2$.

Example 37. The $\sin(1/x)$-curve X defined by

$$X = \{(0, y) \in \mathbb{R}^2\colon\ y \in [-1, 1]\} \cup \{(x, \sin(1/x)) \in \mathbb{R}^2\colon\ x \in (0, 1]\}$$

has property (18) while it does not have property (17).

Proof. In fact, X does not have property (17) since if $\varepsilon = 1$ we can take P as the limit segment $\{(0, y) \in \mathbb{R}^2\colon\ y \in [-1, 1]\}$ of X. Defining, for any $\delta > 0$, a subcontinuum Q of X by $Q = \{(x, y) \in X\colon\ x \in [0, \delta/2]\}$, we see that $H(P, Q) < \delta$, and there is no surjection from P onto Q because P is locally connected, while Q is not.

In order to prove (18) it is enough to verify interiority of $C(f)$ at each $A \in C(X \times Y)$. Equivalently, we have to show that for each $\varepsilon > 0$ there exist a $\delta > 0$ such that $B_{C(X)}(f(A), \delta) \subset C(f)(B_{C(X\times Y)}(A, \varepsilon))$.

Since $X \subset \mathbb{R}^2$, for $i \in \{1, 2\}$ we can consider the ordinary projection $\rho_i : X \to \mathbb{R}$. For each $n \in \mathbb{N}$ put $q_n = 1/(\pi(n - 1/2))$. Let $A \in C(X \times Y)$ be fixed. If $0 \in \rho_1(f(A))$, then we can use Proposition 13 or 14 to show that $C(f)$ is interior at A. Thus we need to consider two significant cases only.

Case 1. $\rho_1(f(A)) = [0, t]$ for some $t \in (0, 1]$. Fix $M \in \mathbb{N}$ such that $q_M, q_{M+1} \in (0, t)$ and $q_M < \varepsilon$. Choose $\delta > 0$ so that if $D \in C(X)$ with $\rho_1(D) = [u, s]$ and $H(f(A), D) < \delta$, then $q_M, q_{M+1} \in (u, s)$ and $\|(s, \sin(1/s)) - (t, \sin(1/t))\| < \varepsilon$. Fix some $D \in B_{C(X)}(f(A), \varepsilon)$ with $\rho_1(D) = [u, s]$.

If $u = 0$, then it is easy to show that there is a surjective mapping $g : f(A) \to D$ such that $\|g(z) - z\| < \varepsilon$ for each $z \in f(A)$. In this subcase, define $B = \{(g(z), y) \in X \times Y\colon\ (z, y) \in A\}$. Then $B \in C(X \times Y)$, $f(B) = D$ and $H(A, B) < \varepsilon$. Thus $D \in C(f)(B_{C(X\times Y)}(A, \varepsilon))$.

If $u > 0$, let $m = \max\{n \in \mathbb{N}\colon\ q_{n+1} \in [u, s]\}$. Define $D_1 = \rho_1^{-1}([u, q_{m+1}])$ and $D_2 = \rho_1^{-1}([q_{m+1}, s])$. Then $D = D_1 \cup D_2$. Define further $A_1 = \rho_1^{-1}([0, q_{m+1}])$ and $A_2 = \rho_1^{-1}([q_{m+1}, t])$. Then $A = A_1 \cup A_2$. Projecting A_1 onto $\rho_1^{-1}([q_{m+1}, q_m])$ and sending A_2 onto D_2 it is easy to verify that there exists a surjective mapping $g : f(A) \to D_2$ such that

(a) $\|g(z) - z\| < \varepsilon$ for each $z \in f(A) = \rho_1^{-1}([0, t])$;

(b) $g((q_{m+1}, \sin(1/q_{m+1}))) = (q_{m+1}, \sin(1/q_{m+1}))$.

Since X is obviously an arc-like continuum, and since each mapping from a continuum into an arc-like continuum is weakly confluent (see, e.g., [18, Theorem 6.16, p. 56]), it follows that the restriction $f|A : A \to X$ is weakly confluent. Since $D_1 \subset A_1 \subset f(A)$, there exists (by weak confluence of $f|A$) a subcontinuum B_0 of A such that $f(B_0) = D_1$.

In this subcase, define $B = B_0 \cup \{(g(z), y) \in X \times Y\colon\ (z, y) \in A\}$. Let a point $(z, y) \in B_0$ be such that $z = f((z, y)) = (q_{m+1}, \sin(1/q_{m+1}))$. Then $(z, y) = (g(z), y)$, which implies that B is a subcontinuum of $X \times Y$. Observe that $f(B) = D$ and $H(A, B) < \varepsilon$. This completes the analysis of Case 1.

Case 2. $\rho_1(f(A)) = 0$. Assume $f(A) = \{0\} \times [-1, 1]$. Other cases can be treated similarly. Let $\delta = \varepsilon$. Take $D \in C(X)$ such that $H(f(A), D) < \varepsilon$. Here we only analyze the case $D = \rho_1^{-1}([0, s])$. The case when D is not of this form can be treated with similar ideas.

Let $m = \min\{n \in \mathbb{N}\colon\ q_n \in [0, s]\}$. For each $n \geqslant m$ consider the mapping $g_n : [-1, 1] \to \rho_1^{-1}([q_{n+1}, q_n])$ defined as the inverse of $\rho_2|\rho_1^{-1}([q_{n+1}, q_n])$. Then g_n is continuous and

$\|g(t) - t\| < \varepsilon$ for each $t \in [-1, 1]$. Let $g_0 : \rho_2(\rho_1^{-1}([q_n, s])) \to \rho_1^{-1}([q_n, s])$ be the inverse of the mapping $\rho_2|\rho_1^{-1}([q_n, s])$. By the argument quoted in Case 1 the mapping $\rho_2 \circ f|A : A \to [-1, 1]$ is weakly confluent, and thereby there exists a subcontinuum B' of A such that $\rho_2(f(B')) = \rho_2(\rho_1^{-1}([q_n, s]))$. Define $B_0 = \{(g_0(\rho_2(z)), y)\colon (z, y) \in B'\}$ and, for each integer $n \geqslant m$ let $B_n = \{(g_n(\rho_2(z)), y)\colon (z, y) \in A\}$. Put

$$B = A \cup B_0 \cup \bigcup\{B_n\colon n \in \mathbb{N} \text{ and } n \geqslant m\}.$$

Since $|g_n(\rho_2(z)) - z| < q_n$ for each $n \in \mathbb{N}$ and $z \in f(A)$, it follows that the continua B_n tend to A. Thus B is compact. Further, for each even integer $n \geqslant m$ take a point of the form $((0, -1), y) \in A$. Then $(g_n(\rho_2((0, -1)), y) = (g_n(-1), y) = (q_n, y)$. Since $(g_{n-1}(\rho_2((0, -1)), y) = (q_n, y)$, it follows that $B_n \cap B_{n-1} \neq \emptyset$. Similarly, $B_m \cap B_0 \neq \emptyset$, and if $n > m$ is odd, then again $B_n \cap B_{n-1} \neq \emptyset$. Hence B is connected. Thus $B \in C(X \times Y)$. Finally, it can easily be shown that $f(B) = D$ and $H(A, B) < \varepsilon$. This finishes the analysis of Case 2, and completes the proof of interiority of $C(f)$ at A. Therefore $C(f)$ is open, i.e., condition (18) is satisfied. The proof is finished. □

Remark 38. There are mappings f quite different than ones described in Theorem 16 such that $C(f)$ is open. Namely in Example 3 of [7] it is shown that there is a mapping f of a solenoid onto itself such that $C(f)$ is light and open, while not a homeomorphism.

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