OPENNESS AND MONOTONEITY OF INDUCED MAPPINGS

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Abstract. It is shown that for locally connected continuum $X$ if the induced mapping $C(f) : C(X) \to C(Y)$ is open, then $f$ is monotone. As a corollary it follows that if the continuum $X$ is hereditarily locally connected and $C(f)$ is open, then $f$ is a homeomorphism. An example is given to show that local connectedness is essential in the result.

All spaces considered in this paper are assumed to be metric. A mapping means a continuous function. We denote by $\mathbb{N}$ the set of all positive integers, and by $\mathbb{C}$ the complex plane. Given a space $S$, a point $c \in S$ and a number $\varepsilon > 0$, we denote by $B_S(c, \varepsilon)$ the open ball in $S$ with center $c$ and radius $\varepsilon$.

A continuum means a compact connected space. Given a continuum $X$ with a metric $d$, we let $2^X$ denote the hyperspace of all nonempty closed subsets of $X$ equipped with the Hausdorff metric $H$ defined by

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\}$$

(see, e.g., [5, (0.1), p. 1 and (0.12), p. 10]). Further, we denote by $C(X)$ the hyperspace of all subcontinua of $X$, i.e., of all connected elements of $2^X$, and by $F_1(X)$ the hyperspace of singletons. The reader is referred to Nadler’s book [5] for needed information on the structure of hyperspaces.

Given a mapping $f : X \to Y$ between continua $X$ and $Y$, we consider mappings (called the induced ones)

$$2^f : 2^X \to 2^Y \text{ and } C(f) : C(X) \to C(Y)$$

defined by

$$2^f(A) = f(A) \text{ for every } A \in 2^X \text{ and } C(f)(A) = f(A) \text{ for every } A \in C(X).$$

A mapping between continua is said to be:

— open provided the image of an open subset of the domain is open in the range;
— monotone provided the point-inverses are connected;
— light provided the point-inverses are zero-dimensional.

The following theorem is the main result of this paper.

1. Theorem. Let a continuum $X$ be locally connected, and a mapping $f : X \to Y$ be such that the induced mapping $C(f) : C(X) \to C(Y)$ is open. Then $f$ is monotone.
Proof. Assume \( f \) satisfies the assumptions of the theorem and that it is not monotone. Let \( p \) and \( q \) be two points of \( X \) such that \( f(p) = f(q) \) that belong to different components of \( f^{-1}(f(p)) \). By continuity of \( f \) there is a positive \( \varepsilon \) such that for every continuum \( L \subset Y \) such that \( f(p) \in L \) and \( H(L, \{f(p)\}) < \varepsilon \) the components of \( f^{-1}(L) \) containing \( p \) and \( q \) respectively are distinct. By local connectedness of \( Y \) there is a continuum \( V \) such that \( f(p) \in \text{int} V \) and \( H(V, \{f(p)\}) < \varepsilon \), i.e., \( V \subset B_Y(f(p), \varepsilon) \). Let \( U_p \) and \( U_q \) be components of \( f^{-1}(V) \) containing \( p \) and \( q \) respectively. Since in locally connected continua components of open sets are open \([4, \S 49, \text{II}, \text{Theorem} 4, \text{p.} 230]\), we conclude that \( p \in \text{int} U_p \) and \( q \in \text{int} U_q \). Let \( \delta > 0 \) be such that \( B_X(p, \delta) \subset U_p \) and \( B_X(q, \delta) \subset U_q \).

Let \( \mathcal{B} \) be an order arc in \( C(Y) \) from \( \{f(p)\} \) to \( Y \) through \( V \). Define \( \mathcal{A} \) as a subset of \( \mathcal{B} \) composed of all elements \( L \in \mathcal{B} \) such that the component of \( f^{-1}(L) \) containing \( p \) is distinct from the component of \( f^{-1}(L) \) containing \( q \). Note that \( V \in \mathcal{A} \) and that if \( L, L' \in \mathcal{B}, L \in \mathcal{A} \) and \( L' \subset L \), then \( L' \in \mathcal{A} \). Thus \( \mathcal{A} \) is a connected subset of \( \mathcal{B} \) containing \( \{f(p)\} \) and \( V \). Since \( \mathcal{B} \setminus \mathcal{A} \) is closed, we see that \( \mathcal{A} \) is an open subset of \( \mathcal{B} \). Let \( Q = \sup \mathcal{A} = \inf(\mathcal{B} \setminus \mathcal{A}) \). Then \( Q \in \text{cl} \mathcal{A} \setminus \mathcal{A} \). Denote by \( P \) the component of \( f^{-1}(Q) \) containing both \( p \) and \( q \). Openness of \( C(f) \) implies that \( f \) is open (see \([3, \text{Theorem} 4.3, \text{p.} 243]\); compare also \([2, \text{Theorem} 3.2]\)), so \( f(P) = Q \) \([6, (7.5), \text{p.} 148]\). We will show that \( C(f)(B_{C(X)}(P, \delta)) \) is not open in \( C(Y) \). So, assume the contrary. Then there is a continuum \( K \in B_{C(X)}(P, \delta) \) with \( f(K) \in \mathcal{A} \). Since \( p,q \in P \) and \( H(P,K) < \delta \), we have \( K \cap U_p \neq \emptyset \neq K \cap U_q \). Then \( U_p \cup K \cup U_q \) is a continuum containing both \( p \) and \( q \), whose image \( f(U_p \cup K \cup U_q) = f(K) \) is in \( \mathcal{A} \), contrary to the definition of \( \mathcal{A} \). The proof is finished. \( \square \)

2. Corollary. Let a continuum \( X \) be hereditarily locally connected, and a mapping \( f : X \to Y \) be such that the induced mapping \( C(f) : C(X) \to C(Y) \) is open. Then \( f \) is a homeomorphism.

Proof. It is enough to show that monotone open mappings on hereditarily locally connected continua are homeomorphisms. Assume the contrary, and let \( y \in Y \) be such that \( f^{-1}(y) \) is a nondegenerate continuum in \( X \). Let \( \{y_n\} \) be an arbitrary sequence converging to \( y \). Then continua \( f^{-1}(y_n) \) tend to \( f^{-1}(y) \), so \( f^{-1}(y) \) is a nondegenerate continuum of convergence, contrary to hereditary local connectedness of \( X \). \( \square \)

3. Example. There are a continuum \( X \) and a mapping \( f : X \to X \) such that \( C(f) : C(X) \to C(X) \) is light and open, but not monotone.

Proof. Let \( S = \{z \in \mathbb{C} : |z| = 1\} \) be the unit circle. For \( n \in \mathbb{N} \), put \( X_n = S \), and let \( \varphi_n : X_{n+1} \to X_n \) be defined by \( \varphi_n(z) = z^3 \). Then \( X = \lim(X_n, \varphi_n) \) is the triadic solenoid. Define \( f : X \to X \) by \( f(\{z_1, z_2, \ldots\}) = \{z_1^2, z_2^2, \ldots\} \), and note that \( f \) is well-defined. It has been proved in \([1, \text{Example} 4.5]\) that the restriction \( C(f)(C(X) \setminus \{X\}) \) is two-to-one and \( C(f)^{-1}(X) \) is a singleton. Thus \( C(f) \) is light and it is not a homeomorphism. We will prove that \( C(f) \) is open. To this aim it is enough to show that the mapping is interior at each point of its domain \([6, \text{p.} 149]\), i.e., that for each \( P \in C(X) \) and for each open neighborhood \( U \) of \( P \) in \( C(X) \) we have \( C(f)(P) \in \text{int} C(f)(U) \). For each \( n \in \mathbb{N} \) let \( f_n : X_n \to X_n \) be defined by \( f_n(z) = z^2 \) (and thus \( f = \lim f_n) \), and let \( \pi_n : X \to X_n \) be the projection. Let \( P \in C(X) \) be a proper subcontinuum of \( X \). Then there exists an index \( n \in \mathbb{N} \) such that \( \pi_{n-1}(P) \) is a proper subcontinuum of \( X_{n-1} \), so \( \pi_n(P) \) is an arc of length less than \( 2\pi/3 \). Let \( U_n \) be an open arc in \( X_n \) containing \( \pi_n(P) \) and having its length still less
than $2\pi/3$. Then the set $V = \{ A \in C(X) : \pi_n(A) \in U_n \}$ is an open neighborhood of $P$ in $X$ such that the restriction $C(f)|V : V \to C(f)(V)$ is a homeomorphism onto the open set $C(f)(U) = \{ A \in C(X) : \pi_n(A) \in f_{n_i}(U_n) \}$ containing $C(f)(P)$. So interiority of $C(f)$ at $P$ is shown in the case $P \neq X$. To prove that $C(f)$ is interior at $X$ consider, for $n \in \mathbb{N}$, the sets $V_n = \{ A \in C(X) : \pi_n(A) = X_n \}$ and note that the family $\{ V_n : n \in \mathbb{N} \}$ is a local base of (closed) neighborhoods of $X$ on $C(X)$. So, it is enough to prove that $C(f)(V_n) \supset V_n+1$. To this end take $A \in V_n+1$, and let $B \in X$ be such that $f(B) = A$. Since $f_{n+1}(\pi_{n+1}(B)) = \pi_{n+1}(f(B)) = \pi_{n+1}(A) = X_{n+1}$, we see that $\pi_{n+1}(B)$ is an arc in $X_{n+1}$ of length at least $\pi$. Thus $\pi_n(B) = \varphi_n(\pi_{n+1}(B)) = X_n$, i.e., $B \in V_n$, whence it follows that $A = f(B) \in C(f)(V_n)$. The proof is then complete.

In connection with Theorem 1 and Example 3 it would be interesting to know if a stronger result is true, namely whether or not the conclusion of Theorem 1 can be deduced from local connectedness of $Y$ only (without assuming local connectedness of $X$). In other words we have the following question.

4. **Question.** Can the assumption of local connectedness of the domain continuum $X$ be relaxed to that of the range continuum $Y$ in Theorem 1?

**References**


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