

Pointwise Smooth Dendroids Have Contractible Hyperspaces

by

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Summary. The theorem stated in the title is proved.

Czuba has asked in [3], (3.11), p. 202 whether pointwise smooth dendroids are hereditarily contractible (the converse is proved in [3], (3.10), p. 202). Since each subcontinuum of a pointwise smooth dendroid is again a pointwise smooth dendroid the problem is equivalent to the problem of contractibility of pointwise smooth dendroids. It is still open. In this paper we only prove that pointwise smooth dendroids have contractible hyperspaces. Perhaps it is a way to solve the problem of contractibility, namely it is reduced to the problem of the existence of a retraction of $C(X)$ onto $F_1(X)$ — the space of all one-point subsets of X .

A continuum is a compact and connected metric space. A dendroid is a hereditarily unicoherent and arcwise connected continuum. If X is a dendroid and $x, y \in X$, then the unique arc from x to y is denoted by xy .

For a given sequence $\{A_n\}$ of subsets of X we use the symbols $Ls A_n$ and $\text{Lim } A_n$ for the upper limit and the limit of the sequence, respectively, as defined in [7], (0.5), p. 4.

A dendroid is called pointwise smooth ([3], (2.2), p. 198) if for each point $x \in X$ there exists a point $p(x) \in X$ such that for each sequence $\{x_n\}$ convergent to x we have $\text{Lim } p(x) x_n = p(x) x$.

Let X be a continuum. Then 2^X and $C(X)$ denote the hyperspaces of all non-empty closed subsets of X and of all subcontinua of X , respectively. A continuous map $\omega: C(X) \rightarrow [0, 1]$ is called a Whitney map if $\omega(\{x\}) = 0$ for all $x \in X$ and if $A \subset B \neq A$ implies $\omega(A) < \omega(B)$. The existence of such mappings for every continuum is shown e.g. in [7], (0.50), p. 24–27. An ordered arc in a hyperspace is an arc ordered by

inclusion. It follows from the definitions that a Whitney map restricted to an ordered arc is one-to-one.

For a given continuum X define a function $K: X \rightarrow 2^X$ by $K(x) = \bigcap \{C \in C(X) : x \in \text{int } C\}$ (see [6], p. 404; cf. [8], p. 373).

We recall now two useful theorems. The first of them is Czub's characterization of pointwise smooth dendroids and the second one is Curtis' characterization of contractibility of hyperspaces.

THEOREM A ([4], Theorem 8, p. 197). *A dendroid is pointwise smooth if and only if for every its point x , the set $K(x)$ is an arc with x as one of its end-points.*

THEOREM B ([2], Theorem 5.4). *Let X be a continuum. Then the hyperspaces 2^X and $C(X)$ are contractible if and only if there exists a lower semicontinuous function $F: X \rightarrow C(C(X))$ such that for every $x \in X$ we have:*

- (1) $\{x\}, X \in F(x)$, and
- (2) if $M \in F(x)$, then there is an arc between $\{x\}$ and M contained in $F(x) \cap C(M)$.

The aim of this paper is to prove the following result.

THEOREM. *For each pointwise smooth dendroid X the hyperspaces 2^X and $C(X)$ are contractible.*

Proof. Define a function $F: X \rightarrow C(C(X))$ by $F(x) = \{M \in C(X) : x \in M, \text{ and either } M \subset K(x) \text{ or } K(x) \subset M\}$. We prove F satisfies the assumptions of Theorem B. Condition (1) is obvious. We verify condition (2). Fix a point $x \in X$ and take $M \in F(x)$. If $M \subset K(x)$, then by Theorem A the continuum M is an arc xy for some $y \in K(x)$ and thus $\{xz : z \in xy\}$ an arc satisfying (2). If $K(x) \subset M$, then by [7], (1.25), p. 74 there are ordered arcs of subcontinua of X from $\{x\}$ to $K(x)$ and from $K(x)$ to M . The union of these two arcs is an ordered arc satisfying (2).

So we have only to show the lower semicontinuity of F . Fix a point $x \in X$ and a continuum $M \in F(x)$, and let a sequence $\{x_n\}$ of points of X converge to x . We have to find continua M_n such that $M_n \in F(x_n)$ and $\text{Lim } M_n = M$. Consider two cases.

Case 1. $K(x) \subset M$. Then by upper semicontinuity of the function K (see [5], Lemma 2, p. 5) we have $\text{Ls } K(x_n) \subset K(x)$, and by the definition of K we have $\text{Ls } xx_n \subset K(x)$, so it is enough to put $M_n = K(x_n) \cup xx_n \cup M$.

Case 2. $M \subset K(x)$. Let, for every natural n , the symbol \mathcal{A}_n denote an ordered arc which is the union of three ordered arcs: from $\{x_n\}$ to $K(x_n)$, from $K(x_n)$ to $K(x_n) \cup xx_n$ and from $K(x_n) \cup xx_n$ to $K(x_n) \cup xx_n \cup K(x)$ (the existence of such arcs follows from [7], (1.25), p. 74).

Let M_n be such a continuum belonging to \mathcal{A}_n that $\omega(M_n) = \omega(M)$. It follows from the definition that either $M_n \subset K(x_n)$ or $K(x_n) \subset M$, so $M_n \in F(x_n)$. We shall show that $\text{Lim } M_n = M$. Let $\{M_{n_k}\}$ be any convergent subsequence of $\{M_n\}$. We prove $\text{Lim } M_{n_k} = M$. Really, observe that (a) $x \in \text{Lim } M_{n_k}$, because $x_n \in M_n$; further, (b) $\text{Lim } M_{n_k} \subset K(x)$, because $\text{Lim } M_{n_k} \subset \text{Ls } M_n \subset \text{Ls } [K(x_n) \cup xx_n \cup K(x)] \subset K(x)$ (the inclusion $\text{Ls } K(x_n) \subset K(x)$ follows from upper semicontinuity of the function K — see [5], Lemma 2, p. 5, and the inclusion $\text{Ls } xx_n \subset K(x)$ follows from the definition of the function K); and (c) $\omega(\text{Lim } M_{n_k}) = \omega(M)$. Note that these three conditions imply $\text{Lim } M_{n_k} = M$ which completes the proof.

COROLLARY. *A dendroid X is pointwise smooth if and only if each subcontinuum Y of X has contractible hyperspaces 2^Y and $C(Y)$.*

In fact, one implication is a consequence of the Theorem and of the heredity of pointwise smoothness for dendroids ([3], (2.3), p. 198). To see the other one assume a dendroid X is not pointwise smooth. Then by [3], (3.7), p. 201 there are subcontinua A and Y with $A \subset Y \subset X$ and such that A is an R^3 -continuum in Y (see [3], (3.4), p. 200 for the definition). Hence by [1], Corollary 4, the hyperspaces 2^Y and $C(Y)$ are not contractible.

We end the paper asking some questions.

QUESTION 1. Characterize those continua that their subcontinua have contractible hyperspaces. Note that there are such continua which are not dendroids, for example the $\sin 1/x$ -curve. In particular: is it true that such continua are exactly those which do not have subcontinua containing R^3 -continua? (For dendroids it follows from [3], (3.9), p. 202 and from the Corollary).

QUESTION 2. Given a pointwise smooth dendroid X , does there exist a retraction $r: C(X) \rightarrow F_1(X)$? ($F_1(X)$ denotes the set of all one-point subsets of X).

REMARK. If the answer to Question 2 is positive, then pointwise smooth dendroids are hereditarily contractible. Really, let $H: C(X) \times [0, 1] \rightarrow C(X)$ be a contraction. Then $h: X \times [0, 1] \rightarrow X$ defined by $h(x, t) = r(H(\{x\}, t))$ is a contraction of X , and since being a pointwise smooth dendroid is a hereditary property (see [3], (2.3), p. 198), the conclusion follows.

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В. Я. Харатоник, Точечно-гладкие дендронды имеют стягиваемые гиперпространства

Доказывается теорема, сформулированная в заголовке.