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GENERAL TOPOLOGY

The Property of Kelley for Fans

by

Janusz J. CHARATONIK and Włodzimierz J. CHARATONIK

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Summary. Fans having the property of Kelley are characterized as ones which can be embedded into a specific subfan of the Cantor fan in a special way. The result implies another characterization of these fans as the limits of inverse sequences of locally connected fans with open bonding mappings.

A *continuum* means a compact connected metric space. It is said to be *hereditarily unicoherent* provided that the intersection of any two its subcontinua is connected. A hereditarily unicoherent and arcwise connected continuum is called a *dendroid*. A point p of a dendroid X is called an *end point* of X provided p is an end point of any arc in X that contains p. The set of all end points of a dendroid X will be denoted by $E(X)$. A point p of a dendroid X is called a *ramification point* of X provided it is the vertex of a simple triod contained in X, i.e., if there are three arcs pa, pb and pc in X having p as the only point of the intersection of any two of them. By a *fan* we understand a dendroid having exactly one ramification point, called the *top* of the fan, and usually denoted by v. A locally connected dendroid is called a *dendrite*. A dendrite is said to be *finite* (*countable*) if it has finitely (countably) many end points. A dendroid X is said to be *smooth* provided there exists a point p in X such that for each sequence of points x_n of X converging to a point x, the sequence of arcs px_n converges to the arc px. It is known that if a fan is smooth, then its top can be taken as a point p in the above definition ([4], Corollary 9, p. 301).

All mappings considered in the paper are assumed to be continuous. A mapping is said to be *open* if it transforms open subsets of the domain onto open subsets of the range. A mapping $f: X \to Y$ between continua X and Y is said to be *confluent* provided for each subcontinuum Q of $f(X) \subset Y$ and for each component K of $f^{-1}(Q)$ we have $f(K) = Q$. It is well known that all open mappings are confluent (sec e.g. [10], Theroem 7.5, p. 148).

Given a point x in a continuum X, let $C(x, X)$ denote the family of all subcontinua of X to which the point x belongs. The union $C(X) = \bigcup\{C(x, X): x \in X\}$, called the *hyperspace of subcontinua* of X, is metrized by the Hausdorff distance dist $(A, B) = \inf\{\varepsilon > 0: A \subset N(B, \varepsilon)$ and $B \subset N(A, \varepsilon)\}$, where $N(A, \varepsilon)$ is the union of the ε-balls about all points of A. A continuum X is said to have the *property of Kelley* provided that for each point x in X, for each sequence of points x_n converging to x and for each continuum $K \in C(x, X)$ there exists a sequence of continua $K_n \in C(x_n, X)$ which has K as its limit.

The property, well known as an important tool in the study of contractibility of hyperspaces, is interesting by its own right, and it has many applications in continua theory. The paper is a continuation of [3], and its aim is to prove two other characterizations of fans having the property of Kelley. The characterizations are expressed in terms of embeddings of the fan into a special subfan of the Cantor fan, and in terms of a possibility of describing the fan as the limit of an inverse sequence of locally connected fans with open bonding mappings.

To formulate the results we need some auxiliary notation and constructions. As usual, N means the set of all positive integers. We denote by F^ω a countable locally connected fan. In other words, F^ω is homeomorphic to the union of countably many straight line segments in the plane every two of which intersect at their common point v only, and such that for each positive integer k length of the k-th segment is equal to $1/k$. So we can write

$$F^\omega = \bigcup\{ve_k: k \in N\}, \tag{1}$$

where v denotes the top of F^ω, $e_k \in E(F^\omega)$ tend to v as k tends to infinity, and ve_k stands for the straight line segment joining v and e_k.

The cone over the Cantor ternary set C is called the *Cantor fan* and is denoted by F_C. For each positive integer k let F_C^k denote a copy of the Cantor fan F_C with $\mathrm{diam}(F_C^k) \leqslant 1/k$, and let F_C^ω be the one-point union of all F_C^k with their tops identified. In other words, each segment ve_k of the locally connected countable fan F^ω is replaced by a copy F_C^k of the Cantor fan, and these copies are situated in the plane in such a way that any two distinct copies have the top v as the only common point. The continuum obtained in this way is just F_C^ω, and thus we can write

$$F_C^\omega = \bigcup\{F_C^k: k \in N\}, \tag{2}$$

where $F_C^i \cap F_C^j = \{v\}$ if $i \neq j$. Therefore F_C^ω is a fan which can be described as one that is homeomorphic to F_C/ve for some fixed $e \in E(F_C)$.

Note that all three considered fans F^ω, F_C and F_C^ω are smooth.

1. Proposition. *The fan F_C^ω is homeomorphic to the limit of an inverse sequence $\{D^n, f^n\}_{n=1}^\infty$ such that each D^n is homeomorphic to F^ω and each bonding mapping $f^n: D^{n+1} \to D^n$ is open.*

Proof. For each $n \in N$ put

$$D^n = F^\omega \times \{0, 1\}^n/\{v\} \times \{0, 1\}^n,$$

where v means the top of F^ω, and note that D^n is homeomorphic to F^ω, and that D^n is naturally embedded into D^{n+1}. Neglecting the embedding mapping for simplification, we may assume $D^n \subset D^{n+1}$ for each n. Next define a mapping $f^n: D^{n+1} \to D^n$ as follows. Put $f^n(v) = v$ (here v stands for the common top of all fans D^n). For a given point $(x, i_1, i_2, \ldots, i_n, i_{n+1})$ of D^{n+1}, where $x \in F^\omega$ and $i_j \in \{0, 1\}$ for $j \in \{1, 2, \ldots, n, n+1\}$, put $f^n((x, i_1, i_2, \ldots, i_n, i_{n+1})) = (x, i_1, i_2, \ldots, i_n)$. It is evident that f^n is an open retraction of D^{n+1} onto D^n which keeps the top v fixed. Lemma 4 of [1], p. 32 says that for each confluent mapping between fans the inverse image is connected of any continuum which contains the top of the range fan. Thereby $(f^n)^{-1}(f^n(v))$ is a connected subset of D^{n+1}, which assures us that the bonding mappings f^n are monotone relative to the top v, according to Theorem 2.3 of [9], p. 720. Since the elements D^n of the inverse sequence are smooth fans, and since the tops v of all fans form a thread of the sequence, we see that all the assumptions of Corollary 5 of [2], p. 146 are satisfied, and thus the limit D of the inverse sequence is a smooth fan. To see that D and F_C^ω are homeomorphic take, for a fixed index k, the straight line segment $ve_k \subset F^\omega$ (see (1)), and consider the sequence of its preimages under bonding mappings f^n. These preimages are finite fans, namely for each positive integer n the mentioned finite fan is just the cone F^n over the set $A^n = \{0, 1\}^n$, i.e., F^n can be understood as $A^n \times [0, 1/k]/A^n \times \{0\}$. Note that for each n we have $(f^n)^{-1}(f^n(F^{n+1})) = F^{n+1}$, and therefore the restrictions $u^n = f^n|F^{n+1}: F^{n+1} \to F^n$ are open by virtue of (7.2) of [10], p. 147. Moreover, $\{F^n, u^n\}_{n=1}^\infty$ is an inverse sequence having the Cantor fan $F_C^k = C \times [0, 1/k]/C \times \{0\}$ as its limit (compare [6], p. 165 and the proof of implication 2°⇒3° for Theorem 1 in [3]). Since F_C^k tend to $\{v\}$ if k tends to infinity, D and F_C^ω are homeomorphic by equality (2), and the proof is complete.

To show the main result of the paper we need an equivalence presented below, which forms a part of Theorem 1 of [3].

2. *Fact.* Let a fan X be given. Then X is smooth and the set $E(X)$ of end points of X is closed if and only if there exists an embedding $h: X \to h(X) \subset F_C$ of X into the Cantor fan such that $h(E(X)) \subset E(F_C)$.

3. THEOREM. *Let a fan X with the top v be given. Then the following conditions are equivalent:*

1° *X has the property of Kelley;*

2° *there exists an embedding $h: X \to h(X) \subset F_C^\omega$ of X into F_C^ω such that $h(E(X)) \subset E(F_C^\omega)$;*

3° *X is the limit of an inverse sequence of locally connected fans with open bonding mappings.*

Proof. 1°⇒2°. It is known (see [3], Theorem 3) that condition 1° is equivalent to

(c) X is smooth and the set $\{v\} \cup E(X)$ is closed.

Since X is smooth, it is embeddable into the Cantor fan (see [1], Theorem 9, p. 27, and [7], Corollary 4, p. 90). Thus without loss of generality we can assume that X is a subset of F_C. If the set $E(X)$ is closed, the conclusion holds by Fact 2 above. So assume $E(X)$ is not closed. However, since $\{v\} \cup E(X)$ is closed (thus compact), its components are continua, and so they are singletons. Consequently, $\{v\} \cup E(X)$ is zero-dimensional, and therefore $E(X)$ can be represented as the union of countably many disjoint closed sets E_k, where $k \in N$, such that

$$\operatorname{Lim} E_k = \{v\}. \tag{3}$$

Putting $X_k = \bigcup \{ve : e \in E_k\}$ for each k, we see that each X_k is closed, so it is a (smooth) fan or an arc, and that $X_i \cap X_j = \{v\}$ for $i \neq j$, and Lim $X_k = \{v\}$. Further, we have $X = \bigcup \{X_k : k \in N\}$. Note that $E(X_k) = E_k$ is closed; applying Fact 2 to each X_k separately we see that for each $k \in N$ there is an embedding $h_k : X_k \to h_k(X_k) \subset F_C^k \subset \bigcup \{F_C^k : k \in N\} = F_C^\omega$ such that

$$h_k(E(X_k)) = h_k(E_k) \subset E(F_C^k). \tag{4}$$

Define $h : X \to h(X) \subset F_C^\omega$ putting $h|X_k = h_k$ for each $k \in N$. Then (3) implies that h is a well-defined embedding of X into F_C^ω. The condition $h(E(X)) \subset E(F_C^\omega)$ follows from the above definition by (3) and (4).

$2° \Rightarrow 3°$. To simplify notation we assume $X \subset F_C^\omega$ and $E(X) \subset E(F_C^\omega)$. According to Proposition 1 there is an inverse sequence of locally connected fans D^n and open bonding mappings $f^n : D^{n+1} \to D^n$ such that $F_C^\omega = \varprojlim \{D^n, f^n\}$. Let $p_n : F_C^\omega \to D^n$ be the projections in this inverse sequence. Define $X^n = p_n(X) \subset D^n$ and $g^n = f^n|X^{n+1}$. Since each locally connected fan is hereditarily locally connected (as a dendrite), the continua X^n are locally connected fans (or arcs, or points). Note that $\{X^n, g^n\}_{n=1}^\infty$ is an inverse sequence having X as its limit ([8], Proposition 2.5.6, p. 137). Since $E(X) \subset E(F_C^\omega)$, it follows that $E(X^n) = E(p_n(X)) \subset E(p_n(F_C^\omega)) = E(D^n)$. Therefore we see directly from the definition that for each n the mapping g^n is interior at each point of its domain ([10], p. 149), which is equivalent to its openness.

$3° \Rightarrow 1°$. This is an immediate consequence of facts that locally connected continua have the property of Kelley and that the property is preserved under the inverse limit operation if bonding mappings are confluent ([5], Theorem 2, p. 190). Thus the proof is complete.

Combining Theorem 3 above with Theorem 3 of [3] we have the following corollary.

4. Corollary. *Let a fan X with the top v be given. Then the following conditions are equivalent:*

1° *X has the property of Kelley;*

2° *X is smooth and for no countable subset of end points $\{e_n : n \in N\}$ is homeomorphic to the union of straight line segments in the plane joining the origin* (0, 0) *with points of the set* $\{(2, 0)\} \cup \{(1, 1/n) : n \in N\}$;

3° *X is smooth and the set* $\{v\} \cup E(X)$ *is closed;*

4° *X is embeddable into* F_C^ω *in such a way that end points of X are mapped to end points of* F_C^ω;

5° *X is the limit of an inverse sequence of locally connected fans with open bonding mappings;*

6° *X is the limit of an inverse sequence of finite fans with confluent bonding mappings.*

INSTITUTE OF MATHEMATICS, WROCŁAW UNIVERSITY, PL. GRUNWALDZKI 2/4, 50-384 WROCŁAW
(INSTYTUT MATEMATYKI, UNIWERSYTET WROCŁAWSKI)

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Я. Е. Харатоник, В. Я. Харатоник, **Свойство Келли для вееров**

В работе характеризуются вееры, обладающие свойством Келли, как вееры, для которых найдется специальное вложение в некоторый подконтинуум канторского веера, и как пределы обратных последовательностей локально связаных вееров с открытыми связующими отображениями.