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ON PROJECTIONS AND LIMIT MAPPINGS OF INVERSE SYSTEMS OF COMPACT SPACES

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For some classes K of mappings we discuss two problems connected with limits of inverse systems: (1) Does the condition that all bonding mappings are in K imply that all projections are in K? (2) Does the condition that all mappings between factor spaces of two given inverse systems are in K imply that the limit mapping between the inverse limit spaces is in K? We answer both these questions in the affirmative for the classes of monotone, of confluent and of weakly confluent mappings of compact spaces, and for some generalizations of these mappings.

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Introduction

Let $S=\{X^\lambda, f^{\lambda\mu}, \Lambda\}$ and $T=\{Y^\sigma, g^{\sigma\tau}, \Sigma\}$ be inverse systems and let $\{\varphi, h^\sigma\}$ be a mapping of S into T. For some classes K of mappings the problems when $h^\sigma \in K$ implies $\varprojlim\{\varphi, h^\sigma\} \in K$ and when $f^{\lambda\mu} \in K$ implies $\pi^\lambda \in K$ were discussed by a number of authors (see e.g. [1, 4, 8]). In the present note we discuss the same problems for more special classes K of mappings, namely for mappings which are monotone, confluent or weakly confluent at some points of their ranges or relative to some points of their domains; as corollaries we get the corresponding results for monotone, for confluent and for weakly confluent mappings.

Preliminaries

Topological spaces considered throughout this paper are assumed to be compact (thus Hausdorff, see [3, p. 165]) and the mappings are continuous. By a continuum we mean a compact connected space. $C(X)$ denotes the hyperspace of subcontinua of X with the Vietoris topology [3, p. 162]. Given a mapping $f: X \to Y$, we define $f_*: C(X) \to C(Y)$ by $f_*(K) = f(K)$ if $K \in C(X)$.

A mapping $f:X\to Y$ from a topological space X onto a topological space Y is said to be

(a) *monotone*, provided that the inverse image of each subcontinuum of Y is connected;

(b) *confluent*, if for each subcontinuum Q of Y each component of $f^{-1}(Q)$ is mapped onto the whole Q [2, p. 213];

(c) *weakly confluent*, if for each subcontinuum Q of Y there exists a component of $f^{-1}(Q)$ which is mapped onto the whole Q [5, p. 98].

Maćkowiak [6, p. 720] has generalized the concept of a monotone mapping saying that a mapping $f:X\to Y$ is monotone relative to a point $p\in X$ if for each subcontinuum Q of Y such that $f(p)\in Q$ the inverse image $f^{-1}(Q)$ is connected. Recall [6, Theorem 2.1, p. 720] that:

(1) A mapping is monotone if and only if it is monotone relative to each point of its domain.

We introduce the following three concepts, which generalize the notion of a confluent mapping. A mapping $f:X\to Y$ from a topological space X onto a topological space Y is said to be

(a) *confluent at a point* $q\in Y$, if for each subcontinuum Q of Y such that $q\in Q$ each component of $f^{-1}(Q)$ is mapped onto the whole Q under f;

(b) *confluent relative to a point* $p\in X$, if for each subcontinuum Q of Y such that $f(p)\in Q$ the component of $f^{-1}(Q)$ containing the point p is mapped onto the whole Q under f;

(c) *weakly confluent at a point* $q\in Y$, if for each subcontinuum Q of Y such that $q\in Q$ there exists a component of $f^{-1}(Q)$ which is mapped onto the whole Q under f.

Note that the following statements are immediate consequences of the above definitions.

(2) Each mapping f monotone relative to a point p is confluent relative to p, and confluent at $f(p)$.

(3) Each mapping f confluent at a point $f(p)$ is confluent relative to p, and each mapping f confluent relative to a point p is weakly confluent at $f(p)$.

(4) A mapping f is confluent at a point $q\in Y$ if and only if it is confluent relative to each point of $f^{-1}(q)$.

(5) A mapping is confluent (weakly confluent) if and only if it is confluent (weakly confluent, respectively) at each point of its range.

(6) A mapping is confluent if and only if it is confluent relative to each point of its domain.

The following notation will be used. $\{X^\lambda, f^{\lambda\mu}, \Lambda\}$ denotes an inverse system of topological spaces X^λ with continuous bonding mappings $f^{\lambda\mu}:X^\mu\to X^\lambda$ for any $\lambda\leqslant\mu$, where λ, $\mu\in\Lambda$, and Λ is a set directed by the relation $\leqslant$. We assume that $f^{\lambda\lambda}$ is the identity, and we denote by $X=\varprojlim\{X^\lambda, f^{\lambda\mu}, \Lambda\}$ the inverse limit space. Further, $\pi^\lambda:X\to X^\lambda$ denotes the projection from the inverse limit space into the

λth factor space. Given a point $p \in X = \varprojlim\{X^\lambda, f^{\lambda\mu}, \Lambda\}$, we put $p^\lambda = \pi^\lambda(p) \in X^\lambda$ and we write $p = \{p^\lambda\}$. Obviously we have

(7) $f^{\lambda\mu}(p^\mu) = p^\lambda$ for any λ, $\mu \in \Lambda$ with $\lambda \leqslant \mu$.

A point $p \in X$, i.e., a system of points $p^\lambda \in X^\lambda$ for $\lambda \in \Lambda$ satisfying (7) is called a thread.

Suppose we are given two inverse systems $S = \{X^\lambda, f^{\lambda\mu}, \Lambda\}$ and $T = \{Y^\sigma, g^{\sigma\tau}, \Sigma\}$. By a mapping of S to T we mean a family $\{\varphi, h^\sigma\}$ consisting of a nondecreasing function $\varphi : \Sigma \to \Lambda$ such that the set $\varphi(\Sigma)$ is cofinal in Λ, and of continuous mappings $h^\sigma : X^{\varphi(\sigma)} \to Y^\sigma$, defined for all $\sigma \in \Sigma$ and such that $g^{\sigma\tau}h^\tau = h^\sigma f^{\varphi(\sigma)\varphi(\tau)}$, i.e., such that the diagram

(8)
$$\begin{tikzcd} X^{\varphi(\sigma)} \arrow[d, "h^\sigma"'] & X^{\varphi(\tau)} \arrow[l, "f^{\varphi(\sigma)\varphi(\tau)}"'] \arrow[d, "h^\tau"] \\ Y^\sigma & Y^\tau \arrow[l, "g^{\sigma\tau}"] \end{tikzcd}$$

is commutative for any σ, $\tau \in \Sigma$ satisfying $\sigma \leqslant \tau$. Any mapping of S to T induces a continuous mapping of $X = \varprojlim S$ to $Y = \varprojlim T$, called the limit mapping induced by $\{\varphi, h^\sigma\}$ and denoted by $h = \varprojlim\{\sigma, h^\sigma\}; X \to Y$ (see [3, p. 138 and 139]).

PART A: PROJECTIONS

In this part we discuss the following problem. Given a class K of mappings, does the condition that all bonding mappings $f^{\lambda\mu}$ are in K imply that all projections π^λ are in K, too? We show an affirmative answer to this question in case when K is the class of monotone, confluent and weakly confluent mappings at some point of the range or relative to some point of the domain. Moreover, a slightly stronger result is obtained: for some fixed element α of the directed set, the condition $f^{\alpha\lambda} \in K$ for all λ with $\alpha \leqslant \lambda$ implies $\pi^\alpha \in K$. As corollaries we get corresponding theorems for monotone, confluent and weakly confluent mappings.

In what follows Λ always denotes a directed set and α is an arbitrary but fixed element of Λ. Further, in the whole Part A, we consider an inverse system $\{X^\lambda, f^{\lambda\mu}, \Lambda\}$ of compact spaces X^λ and we denote by X its inverse limit. Since the set $\{\lambda \in \Lambda : \alpha \leqslant \lambda\}$ is cofinal in Λ, we can assume by [3, 2.5.11, p. 140] without loss of generality that $\alpha \leqslant \lambda$ for all $\lambda \in \Lambda$.

We start with the following lemma.

Lemma 1. *Let α be the only first element of a directed set Λ and let $X = \varprojlim\{X^\lambda, f^{\lambda\mu}, \Lambda\}$. For $Q \subset X^\alpha$ put $Q^\lambda = (f^{\alpha\lambda})^{-1}(Q)$. Then*

(9) $(\pi^\alpha)^{-1}(Q) = \varprojlim\{Q^\lambda, f^{\lambda\mu} \mid Q^\mu, \Lambda\}$.

Proof. Observe that $f^{\lambda\mu}(Q^{\mu}) \subset Q^{\lambda}$, whence the considered inverse system in the right member of (9) is well defined. Now both inclusions which form equality (9) are direct consequences of the definition of the inverse limit.

1. Monotone mappings

The main result of this section is:

Theorem 1. *If there exists a thread* $p = \{p^{\lambda}\}$ *such that for each* $\lambda \in \Lambda$ *with* $\alpha \leqslant \lambda$ *the bonding mapping* $f^{\alpha\lambda}$ *is monotone relative to* p^{λ}, *then the projection* π^{α} *is monotone relative to* p.

Proof. Take a continuum Q in X^{α} which contains the point $p^{\alpha} = \pi^{\alpha}(p)$. We have to show that the set $(\pi^{\alpha})^{-1}(Q)$ is connected. To this effect put $Q^{\lambda} = (f^{\alpha\lambda})^{-1}(Q)$ for $\lambda \in \Lambda$. Note that $p^{\lambda} \in Q^{\lambda}$, and since the mapping $f^{\alpha\lambda}$ is monotone relative to the point p^{λ}, we conclude that Q^{λ} is a continuum for each $\lambda \in \Lambda$. Therefore $(\pi^{\alpha})^{-1}(Q)$ is the inverse limit of continua Q^{λ} according to (9) of Lemma 1, so it is a continuum by [3, Theorem 6.1.18, p. 436].

Corollary 1. *If there exists a thread* $p = \{p^{\lambda}\}$ *such that the bonding mappings* $f^{\lambda\mu}$ *are monotone relative to* p^{μ} *for all* λ, $\mu \in \Lambda$ *with* $\lambda \leqslant \mu$, *then the projections* π^{λ} *are monotone relative to* p *for all* $\lambda \in \Lambda$.

From Corollary 1 and (1) we get (see [1, Lemma 4.2, p. 241]).

Corollary 2. *If all bonding mappings* $f^{\lambda\mu}$ *are monotone, then all projections* π^{λ} *are monotone, too.*

2. Confluent mappings

Similar results can be obtained for the class of confluent mappings. Namely we have:

Theorem 2. *If there exists a thread* $p = \{p^{\lambda}\}$ *such that, for each* $\lambda \in \Lambda$ *with* $\alpha \leqslant \lambda$, *the mapping* $f^{\alpha\lambda}$ *is confluent relative to* p^{λ}, *then the projection* π^{α} *is confluent relative to* p.

Proof. Take a continuum Q in X^{α} which contains the point $p^{\alpha} = \pi^{\alpha}(p)$. We shall show that there exists a continuum K containing p and contained in $(\pi^{\alpha})^{-1}(Q)$ that is mapped onto the whole Q under π^{α} (note that the existence of such K finishes the proof). Let Q^{λ} be the component of $(f^{\alpha\lambda})^{-1}(Q)$ that contains the point

p^λ. Note that $f^{\lambda\mu}(Q^\mu) \subset Q^\lambda$ for each λ and μ in Λ satisfying $\lambda \leqslant \mu$, so $\{Q^\lambda, f^{\lambda\mu} | Q^\mu, \Lambda\}$ is an inverse system. Indeed, $p^\lambda \in f^{\lambda\mu}(Q^\mu) \cap Q^\lambda$, and obviously $f^{\lambda\mu}(Q^\mu) \subset (f^{\alpha\lambda})^{-1}(Q)$, thus the inclusion follows by the definition of Q^λ as the component of $(f^{\alpha\lambda})^{-1}(Q)$ containing the point p^λ. Now put $K = \varprojlim\{Q^\lambda, f^{\lambda\mu} | Q^\mu, \Lambda\}$ and observe that K is a continuum as the inverse limit of an inverse system of continua Q^λ [3, Theorem 6.1.18, p. 436]. Since $p^\alpha = f^{\alpha\lambda}(p^\lambda) \in Q$ and since each $f^{\alpha\lambda}$ is confluent relative to p^λ, we have $f^{\alpha\lambda}(Q^\lambda) = Q$, which shows that the mappings $f^{\alpha\lambda} | Q^\lambda : Q^\lambda \to Q^\alpha$ are onto, so by [1, 2.6, p. 235] the projection $\pi^\alpha | K : K \to Q$ is onto. Thus the proof is complete.

Corollary 3. *If there is a point $q \in X^\alpha$ such that for each $\lambda \in \Lambda$, with $\alpha \leqslant \lambda$, the mapping $f^{\alpha\lambda}$ is confluent at the point q, then the projection π^α is confluent at q.*

Indeed, we ought to show by (4) that π^α is confluent relative to each point of $(\pi^\alpha)^{-1}(q)$. Applying (9) of Lemma 1 with $\{q\}$ in place of Q we have

$$(\pi^\alpha)^{-1}(q) = \varprojlim\{(f^{\alpha\lambda})^{-1}(q), f^{\lambda\mu} | (f^{\alpha\mu})^{-1}(q), \Lambda\}.$$

Using (4) once more, the conclusion holds by virtue of Theorem 2.

Theorem 2 and (6) or Corollary 3 and (5) imply

Corollary 4. *If for each $\lambda \in \Lambda$ with $\alpha \leqslant \lambda$ the bonding mapping $f^{\alpha\lambda}$ is confluent, then the projection π^α is confluent.*

As immediate consequences of Theorem 2 and Corollary 3 we get the next two corollaries.

Corollary 5. *If there exists a thread $p = \{p^\lambda\}$ such that for each λ, $\mu \in \Lambda$ with $\lambda \leqslant \mu$ the bonding mapping $f^{\lambda\mu}$ is confluent relative to p^λ, then the projection π^λ is confluent relative to p for each $\lambda \in \Lambda$.*

Corollary 6. *If for each $\lambda \in \Lambda$ there is a point $q^\lambda \in X^\lambda$ such that for each $\mu \in \Lambda$ with $\lambda \leqslant \mu$ the bonding mapping $f^{\lambda\mu}$ is confluent at q^λ, then for each $\lambda \in \Lambda$ the projection π^λ is confluent at q^λ.*

Corollary 5 and (6) or Corollary 6 and (5) imply

Corollary 7. *If all bonding mappings $f^{\lambda\mu}$ are confluent, then all projections π^λ also are confluent.*

3. Weakly confluent mappings

Analogous conclusions as for monotone and confluent mappings can also be obtained for the class of weakly confluent (and related) mappings. However, the methods of the proof of the main result of this section are different from previous ones: they rely upon some hyperspace techniques.

We start with

Lemma 2. *If for each* $\lambda \in \Lambda$, *the non-empty set* $Z^\lambda \subset X^\lambda$ *is compact and if for each* $\lambda, \mu \in \Lambda$ *with* $\lambda \leqslant \mu$ *we have* $f^{\lambda\mu}(Z^\mu) \subset Z^\lambda$, *then there exists a thread* $\{z^\lambda\}$ *such that* $z^\lambda \in Z^\lambda$ *for each* $\lambda \in \Lambda$.

Proof. Under assumptions of the lemma the sets Z^λ and the mappings $f^{\lambda\mu} | Z^\mu : Z^\mu \to Z^\lambda$ form an inverse system of compact spaces. Its limit space is compact and non-empty [1, 2.5, p. 235].

Theorem 3. *If there is a point* $q \in X^\alpha$ *such that for each* $\lambda \in \Lambda$ *with* $\alpha \leqslant \lambda$ *the mapping* $f^{\alpha\lambda}$ *is weakly confluent at the point* q, *then the projection* π^α *is weakly confluent at* q.

Proof. Let $Q \subset X^\alpha$ be a continuum with $q \in Q$. It is to be shown that there exists a continuum $K \subset X$ such that $\pi^\alpha(K) = Q$. To this end define, for each $\lambda, \mu \in \Lambda$ with $\lambda \leqslant \mu$, the induced mapping $f_*^{\lambda\mu} : C(X^\mu) \to C(X^\lambda)$ by $f_*^{\lambda\mu}(P) = f^{\lambda\mu}(P)$ if $P \in C(X^\mu)$, and put $Z^\lambda = \{P \in C(X^\lambda) : f^{\alpha\lambda}(P) = Q\}$ for all $\lambda \in \Lambda$ satisfying $\alpha \leqslant \lambda$. Hence Z^λ is a compact subset of $C(X^\lambda)$, and it is non-empty by weak confluence of $f^{\alpha\lambda}$. It is easy to see that the sets Z^λ satisfy the conditions of Lemma 2 applied to the inverse system $\{C(X^\lambda), f_*^{\lambda\mu}, \Lambda\}$. Hence there exists a point $K = \{K^\lambda\} \in \varprojlim\{C(X^\lambda), f_*^{\lambda\mu}, \Lambda\}$ such that $K^\lambda \in Z^\lambda$ for all $\lambda \in \Lambda$. Thus by [7, (5), p. 172 and Remark 1.170 p. 174] we have $K = \varprojlim\{K^\lambda, f^{\lambda\mu} | K^\mu, \Lambda\}$. Since the bonding mappings $f^{\lambda\mu} | K^\mu$ are onto, then by [1, 2.6, p. 235] the projection $\pi^\alpha : K \to Q$ is onto. This completes the proof.

As previously, using (5), we get some corollaries.

Corollary 8. *If for each* $\lambda \in \Lambda$ *with* $\alpha \leqslant \lambda$ *the bonding mappings* $f^{\alpha\lambda}$ *are weakly confluent, then the projection* π^α *is weakly confluent.*

Corollary 9. *If for each* $\lambda \in \Lambda$ *there is a point* $q^\lambda \in X^\lambda$ *such that for each* $\mu \in \Lambda$ *with* $\lambda \leqslant \mu$ *the bonding mappings* $f^{\lambda\mu}$ *are weakly confluent at* q^λ, *then for each* $\lambda \in \Lambda$ *the projection* π^λ *is weakly confluent at* q^λ.

Corollary 10. *If all bonding mappings* $f^{\lambda\mu}$ *are weakly confluent, then all projections* π^λ *also are weakly confluent.*

PART B: LIMIT MAPPINGS

The present part of the paper is devoted to mappings between two inverse systems. The problem we are investigating in is the following. Given a class K of mappings, does the condition that all mappings h^σ between the particular factor spaces of the two inverse systems are in K imply that the limit mapping between the inverse limit spaces is in K, too? We prove that an answer to this question is yes if K is the class of monotone, confluent and weakly confluent mappings at some point of the range or relative to some point of the domain. As corollaries we get corresponding theorems for monotone, confluent and weakly confluent mappings.

In the whole Part B we consider two directed sets Λ and Σ and a non-decreasing function $\varphi:\Sigma\to\Lambda$ such that the set $\varphi(\Sigma)$ is cofinal in Λ; the sets Λ and Σ serve as sets of indices for two inverse systems $S=\{X^\lambda, f^{\lambda\mu}, \Lambda\}$ and $T=\{Y^\sigma, g^{\sigma\tau}, \Sigma\}$, with compact spaces X^λ and Y^σ. Recall that X and Y mean the inverse limits of S and T respectively, and denote by $\pi^\lambda: X\to X^\lambda$ and $\rho^\sigma: Y\to Y^\sigma$ the corresponding projections. Further, it is assumed in Part B that we are given a mapping $\{\varphi, h^\sigma\}$ from S to T such that diagram (8) commutes, and we put $h=\varprojlim\{\varphi, h^\sigma\}: X\to Y$. To simplify denotations, we do not distinguish the inverse limits $\varprojlim\{X^{\varphi(\sigma)}, f^{\varphi(\sigma)\varphi(\tau)}, \Sigma\}$ and $\varprojlim\{X^\lambda, f^{\lambda\mu}, \Lambda\}$, which obviously are homeomorphic under our assumptions, and we denote both of them by X.

As in Part A we consecutively discuss monotone, confluent and weakly confluent mappings.

1. Monotone mappings.

The main result of this section is

Theorem 4. *If there exists a thread $p=\{p^{\varphi(\sigma)}\}$ in X such that for each $\sigma\in\Sigma$ the mapping h^σ from $X^{\varphi(\sigma)}$ onto Y^σ is monotone relative to $p^{\varphi(\sigma)}$, then the limit mapping $h: X\to Y$ is monotone relative to p.*

Proof. Take a continuum Q in Y with $h(p)\in Q$. We have to prove that $h^{-1}(Q)$ is connected. Putting $Q^\sigma=\rho^\sigma(Q)$ we have $h^\sigma(p^{\varphi(\sigma)})\in Q^\sigma$. Since the mappings h^σ are monotone relative to $p^{\varphi(\sigma)}$ for $\sigma\in\Sigma$, the sets $(h^\sigma)^{-1}(Q^\sigma)$ are continua. Thus commutativity of diagram (8) leads to the equality

$$f^{\varphi(\sigma)\varphi(\tau)}((h^\tau)^{-1}(Q^\tau))=(h^\sigma)^{-1}(Q^\sigma),$$

which shows that the continua $(h^\sigma)^{-1}(Q^\sigma)$ form an inverse system with the mappings $f^{\varphi(\sigma)\varphi(\tau)}|(h^\tau)^{-1}(Q^\tau)$, where $\sigma\leqslant\tau$. By the definition of $h=\varprojlim\{\varphi, h^\sigma\}$ we conclude from [3, 2.5.11, p. 140] that $h^{-1}(Q)$ is homeomorphic to

$$\varprojlim\{(h^\sigma)^{-1}(Q^\sigma), f^{\varphi(\sigma)\varphi(\tau)}|(h^\tau)^{-1}(Q^\tau), \Sigma\},$$

which is a continuum as the inverse limit of an inverse system of continua [3, 6.1.18, p. 436].

It follows from Theorem 4 by (1) (see [4, Theorem 5, p. 58] and [8, Theorem 10, p. 69]) that

Corollary 11. *If for each* $\sigma \in \Sigma$ *the mapping* h^σ *from* $X^{\varphi(\sigma)}$ *onto* Y^σ *is monotone, then the limit mapping* $h: X \to Y$ *is monotone.*

2. Confluent mappings

Theorem 5. *If there exists a thread* $p = \{p^{\varphi(\sigma)}\}$ *in* X *such that for each* $\sigma \in \Sigma$ *the mapping* h^σ *from* $X^{\varphi(\sigma)}$ *onto* Y^σ *is confluent relative to* $p^{\varphi(\sigma)}$, *then the limit mapping* $h: X \to Y$ *is confluent relative to* p.

Proof. Take a continuum Q in Y with $h(p) \in Q$. We shall show that there exists a continuum $K \subset X$ such that $p \in K \subset h^{-1}(Q)$ and $h(K) = Q$, which suffices to close the proof (as in the proof of Theorem 2). Put $Q^\sigma = \rho^\sigma(Q)$, and let $K^{\varphi(\sigma)}$ be the component of $(h^\sigma)^{-1}(Q^\sigma)$ containing the point $p^{\varphi(\sigma)}$. To prove that the continua $K^{\varphi(\sigma)}$ form an inverse system with the mappings $f^{\varphi(\sigma)\varphi(\tau)}|K^{\varphi(\tau)}$ for σ, $\tau \in \Sigma$ and $\sigma \leq \tau$, we ought to show that

$$f^{\varphi(\sigma)\varphi(\tau)}(K^{\varphi(\tau)}) \subset K^{\varphi(\sigma)}.$$

Indeed, $f^{\varphi(\sigma)\varphi(\tau)}(K^{\varphi(\tau)})$ is a continuum containing the point $p^{\varphi(\sigma)}$, whose image under h^σ is the whole Q^σ (by commutativity of diagram (8)), so it is contained in the component $K^{\varphi(\sigma)}$. To finish the proof put $K = \varprojlim\{K^{\varphi(\sigma)}, f^{\varphi(\sigma)\varphi(\tau)}|K^{\varphi(\tau)}, \Sigma\}$, and observe K is a continuum which is mapped onto Q by [3, 3.2.14, p. 189], because all mappings h^σ map $K^{\varphi(\sigma)}$ onto Q^σ by their confluence relative to $p^{\varphi(\sigma)}$.

By virtue of (4), Theorem 5 implies

Corollary 12. *If there exists a thread* $q = \{q^\sigma\}$ *in* Y *such that for each* $\sigma \in \Sigma$ *the mapping* h^σ *from* $X^{\varphi(\sigma)}$ *onto* Y^σ *is confluent at* q^σ, *then the limit mapping* $h: X \to Y$ *is confluent at* q.

Theorem 5 and (6) or Corollary 12 and (5) imply

Corollary 13. *If for each* $\sigma \in \Sigma$ *the mapping* h^σ *from* $X^{\varphi(\sigma)}$ *onto* Y^σ *is confluent, then the limit mapping* $h: X \to Y$ *is confluent.*

3. Weakly confluent mappings

Similarly to previous results, the following theorem holds.

Theorem 6. *If there exists a thread* $q = \{q^\sigma\}$ *in* Y *such that for each* $\sigma \in \Sigma$ *the mapping* h^σ *from* $X^{\varphi(\sigma)}$ *onto* Y^σ *is weakly confluent at* q^σ, *then the limit mapping* $h: X \to Y$ *is weakly confluent at* q.

Proof. Take a continuum Q such that $q \in Q \subset Y$ and put $Q^\sigma = \rho^\sigma(Q)$ for each $\sigma \in \Sigma$. We have to find a continuum $K \subset X$ such that $h(K) = Q$. In this aim put $Z^\sigma = \{P \in C(X^{\varphi(\sigma)})\colon h^\sigma(P) = Q^\sigma\}$ for each $\sigma \in \Sigma$. Hence Z^σ is a compact subset of $C(X^{\varphi(\sigma)})$, and it is non-empty by weak confluence of h^σ. It can be easily verified that the sets Z^σ satisfy all hypotheses of Lemma 2 applied to the inverse system $\{C(X^{\varphi(\sigma)}), f_*^{\varphi(\sigma)\varphi(\tau)}, \Sigma\}$. Hence there exists a point

$$K = \{K^\sigma\} \in \varprojlim\{C(X^{\varphi(\sigma)}), f_*^{\varphi(\sigma)\varphi(\tau)}, \Sigma\}$$

such that $K^\sigma \in Z^\sigma$ for all $\sigma \in \Sigma$. Thus by [7, (5), p. 172 and Remark 1.170, p. 174] we have $K = \varprojlim\{K^\sigma, f^{\varphi(\sigma)(\tau)} | K^\tau, \Sigma\}$. Since $h^\sigma(K^\sigma) = Q^\sigma$, hence $h(K) = Q$ and we are done.

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