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QUESTIONS ON INDUCED UNIVERSAL MAPPINGS

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ABSTRACT. A mapping between topological spaces is universal if it has a coincidence point with any mapping between the spaces. Given a mapping f between continua X and Y we denote by 2^f (by $C(f)$) the induced mappings between hyperspaces of all nonempty compact subsets (of all nonempty subcontinua) of X and Y, respectively. Implications are discussed from universality of one of these three mappings to universality of the other ones. Some examples are constructed and questions are asked.

A *continuum* means a compact, connected metric space, and a *mapping* means a continuous transformation. A mapping $f : X \to Y$ between topological spaces X and Y is said to be *universal* provided that it has a coincidence with every mapping from X into Y, or — more precisely — provided that for every mapping $g : X \to Y$ there exists a point $x \in X$ such that $f(x) = g(x)$. The concept of a universal mapping has been introduced in [5, p. 603] by W. Holsztyński.

Given a continuum X with a metric d, we let 2^X to denote the hyperspace of all nonempty closed subsets of X equipped with the Hausdorff metric H defined by

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\}$$

(see e.g. [10, (0.1), p. 1 and (0.12), p. 10]). Further, we denote by $C(X)$ the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X, and by $F_1(X)$ the hyperspace of singletons. Given a mapping $f : X \to Y$ between continua X and Y, we consider mappings (called the *induced* ones)

$$2^f : 2^X \to 2^Y \quad \text{and} \quad C(f) : C(X) \to C(Y)$$

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defined by

$$2^f(A) = f(A) \text{ for every } A \in 2^X \quad \text{and} \quad C(f)(A) = f(A) \text{ for every } A \in C(X).$$

The reader is referred to Nadler's book [10] for needed information on the structure and properties of hyperspaces.

The aim of this note is to discuss necessary or sufficient conditions (concerning spaces as well as mappings) under which implications between the following assertions hold, in which X and Y are continua:

(a) $f : X \to Y$ is universal;
(b) $2^f : 2^X \to 2^Y$ is universal;
(c) $C(f) : C(X) \to C(Y)$ is universal.

The following statement is obvious.

1. Statement. *Each universal mapping is surjective.*

Questions concerning universal mappings are related to fixed point questions. If there exists a universal mapping from X onto Y, then for any mapping $h : Y \to Y$ defining $g : X \to Y$ by $g = h \circ f$ we get, by the universality of f, a point $x \in X$ such that $g(x) = h(f(x)) = f(x)$. Putting $y = f(x) \in Y$ we get $h(y) = y$, so the mapping h has a fixed point. Thus we have the following (well known) result.

2. Statement. (i) *A universal mapping from X onto Y can exist only if Y has the fixed point property.* (ii) *The identity $f : Y \to Y$ is universal if and only if Y has the fixed point property.*

Let a number $\varepsilon > 0$ be given. A mapping $f : X \to Y$ between continua is called an *ε-mapping* provided that $\operatorname{diam} f^{-1}(y) < \varepsilon$ for each point $y \in Y$. A continuum is said to be an *arc-like* provided that for each $\varepsilon > 0$ there is an ε-mapping from X onto the closed unit interval $[0, 1]$. This is equivalent to the definition of a snake-like continuum in [8, §48, X, p. 224]. The following result is proved in [6, Theorem 3, p. 437].

3. Statement (Holsztyński). *Each mapping from a connected space onto an arc-like continuum is universal.*

Arc-likeness is essential in this result: one can easily define a mapping from an arc onto a simple triod which is not universal.

A mapping $f : X \to Y$ between continua is said to be *weakly confluent* provided that for each subcontinuum Q of the range Y there exists a subcontinuum C of the domain X such that $f(C) = Q$. The following result is shown in [14, Theorem 4, p. 236].

4. Statement (Read). *Each mapping from a continuum onto an arc-like continuum is weakly confluent.*

The next statement is a consequence of the definitions, see [10, (0.49.1), p. 24].

5. Statement. *Let a mapping $f : X \to Y$ be surjective. Then the induced mapping $C(f)$ is surjective if and only if f is weakly confluent.*

Since condition (c) implies that the induced mapping $C(f)$ is surjective (see Statement 1), then Statement 5 implies the next one.

6. Statement. *If $C(f)$ is universal, then f is weakly confluent.*

Let us quote one more result that is related to mappings onto arc-like continua. It is shown in [11, Theorem 2.11, p. 226].

7. Statement (Nadler). *For each mapping $f : X \to Y$ from a continuum X onto an arc-like continuum Y the induced mapping $C(f)$ is universal.*

Statements 3 and 7 imply that if the continuum Y is arc-like, then assertions (a) and (c) are satisfied. Thus the question concerning (b) arises in a natural way.

8. Question. Let $f : X \to Y$ be a mapping from a continuum X onto an arc-like continuum Y. Is then the induced mapping 2^f universal?

A partial mapping to Question 8 can be obtained from the following result (see [12, Theorem 2.3, p. 752].

9. Statement (Nadler). *For each monotone mapping $f : X \to Y$ from a continuum X onto a locally connected continuum Y the two induced mappings 2^f and $C(f)$ are universal.*

The above statement can be used as an argument in showing that neither (b) nor (c) (nor both) imply (a). Namely we have the following example.

10. Example. *Let S^1 be the unit circle, and let $f : S^1 \to S^1$ be the identity mapping. Then both 2^f and $C(f)$ are universal, while f is not.*

Proof. Indeed, the two induced mappings are universal by Statement 9, while f (as well as any mapping onto S^1) is not universal by Statement 2.

Example 10 motivates the following questions.

11. Questions. Let $f : X \to Y$ be a mapping from a continuum X onto a continuum Y which has the fixed point property. Under what conditions universality of either 2^f or $C(f)$ (or both) implies universality of f?

Now we will show that (a) implies neither (b) nor (c).

12. Example. *The continuum X which is the the union of a disk and a spiral approximating its boundary has the fixed point property, while both the hyperspaces 2^X and $C(X)$ do not have the property.*

Proof. Let $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ be the unit disk in the plane $\mathbb{R}^2$, let $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be its boundary, and let $L = \{(1 + 1/t)e^{it} : t \in [1, +\infty)\} \subset \mathbb{R}^2$ be the spiral approximating S. Put $X = D \cup L$ and $Y = S \cup L$. Then X has the fixed point property, see [1, Theorem 12, p. 123]. To show that 2^X does not have the fixed point property note that $2^S \subset 2^D$, and since 2^S is homeomorphic to the Hilbert cube, [3, Theorem 1, p. 927] and [4, Theorem 3.2, p. 21], which is an absolute retract, [8, §53, III, Theorem 7, p. 341], there exists a retraction $r_0 : 2^D \to 2^S$. Define a retraction $r : 2^X \to 2^Y$ by $r(A) = (A \cap L) \cup r_0(A \cap D)$ for each $A \in 2^X$. It is shown in [2, Theorem, p. 32] that there exists a retraction $r' : 2^Y \to C(Y) \subset 2^Y$. Further, the hyperspace $C(Y)$ is homeomorphic to the cone $\text{Cone}\,(Y)$ (see [15, Theorem 8, p. 283]; also [9, Theorem 1.1, p. 322] and compare [10, Theorem (8.23), p. 322]). Let $h : C(Y) \to \text{Cone}\,(Y)$ be a homeomorphism. Thus the composition $h \circ r' \circ r$ maps 2^X onto $\text{Cone}\,(Y)$. Since retractions preserve the fixed point property, [8, §53, III, Theorem 12, p. 343], and since $\text{Cone}\,(Y)$ does not have the property, [7, Proof of Theorem 2.6, p. 40] and compare [1, Theorem 21, p. 129], it follows that 2^X does not have the fixed point property.

Finally $C(X)$ does not have the fixed point property as well, see [13, Theorem, p. 256] and compare [10, Theorem (7.3), p. 292]. The proof is complete.

Combining Statement 2 with Example 12 we get a corollary.

13. Corollary. *Let X be the union of the disk and the spiral as in Example* 12. *Then the identity $f : X \to X$ is universal, while 2^f and $C(f)$ are not.*

Note that the continuum X of Example 12, containing the disk D, is two-dimensional. So, the next question arises.

14. Questions. (1) Does there exist a one-dimensional continuum X and a universal mapping $f : X \to X$ such that $C(f)$ is not universal? If yes, (2) can f with the above property be the identity? (Compare also [10, Question (7.8.1), p. 297].)

15. Questions. (1) Does there exist a one-dimensional continuum X and and a universal mapping $f : X \to X$ such that 2^f is not universal? If yes, (2) can f with the above property be the identity?

Note that the question (2) above is equivalent, by Statement 2, to the following one.

16. Question. Does there exist a one-dimensional continuum X such that X has, while 2^X does not have, the fixed point property?

17. Questions. (1) Does (b) imply (c)? (2) Does (c) imply (b)?

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