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MONOTONE-OPEN MAPPINGS OF RATIONAL CONTINUA

JANUSZ J. CHARATONIK AND WŁODZIMIERZ J. CHARATONIK

Abstract

It is proved that each monotone-open nonconstant mapping defined on a rational continuum is a homeomorphism. As a consequence it follows that each rational continuum which is homogeneous with respect to monotone-open mappings is a simple closed curve.

1. Introduction

Very recently a number of new results concerning continuous monotone decompositions of continua appeared in the literature (see e.g. [8], [9], [10], [11], [12]). Existence of such decompositions, which is equivalent to the existence of a monotone and open mapping defined on the considered continuum, is related to homogeneity with respect to the class of monotone-open mappings. Recall that, given a class $\mathfrak{M}$ of mappings between continua (which contains homeomorphisms and is closed with respect to compositions), a topological space is said to be *homogeneous with respect to* $\mathfrak{M}$ provided that for very two points in the space there is a mapping in $\mathfrak{M}$ of the space onto itself that maps one of the points to the other. In [2,Theorem 5, p. 131] the first named author has shown that the Sierpiński universal plane curve is homogeneous with respect to monotone mappings, and asked if it is homogeneous with respect to open ones ([2, Problem 1, p. 130 and Problem 3, p. 132]). An affirmative answer to this question was given even in so strong form that the curve is homogeneous with respect to mappings which are monotone and open simultaneously, [8, Corollary 25] and [12, Theorem 15]. Since open mappings between locally compact spaces do not increase Menger-Urysohn order of a point (see e.g. [14, Corollary (7.31), p. 147]), homogeneity of a continuum with respect to open mappings forces the continuum to have the same Menger-Urysohn order of a point at all points of the continuum, see [3, Proposition 2, p. 492]. It is an old result of P. S. Urysohn saying that if all points of a continuum are of the same order $\mathfrak{n}$, then this order can take only four values, namely $\mathfrak{n} \in \{2, \omega, \aleph_0, \mathfrak{c}\}$, see [13, Chapter VI, Section 2, p. 105] and compare [3, Proposition 4, p. 493]. If locally connected plane curves are under consideration, then the examples of such continua with the same order of all points are known. Namely, for

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$\mathfrak{n} = 2$ we have the simple closed curve that is homogeneous (with respect to homeomorphisms) and which is known to be the only one homogeneous locally connected plane continuum. For $\mathfrak{n} = \mathfrak{c}$ the Sierpiński universal plane curve is such, and — as it was recalled above — it is homogeneous with respect to monotone-open mappings, [8, Corollary 25] and [12, Theorem 15]. For $\mathfrak{n} \in \{\omega, \aleph_0\}$ locally connected plane continua $X(\mathfrak{n})$ having all points of the same order $\mathfrak{n}$ have been constructed by Urysohn ([13, Chapter VI, Sections 6-8, and 9-10, pp. 109-115]; compare [3, Example 5, p. 494 and Example 23, p. 496]; for $\mathfrak{n} = \omega$ see also [7, Chapter VIII, Section 5, p. 279]), but neither their intrinsic topological characterizations are known nor homogeneity properies were investigated. In particular, questions asked in [3, Questions 47, p. 502] as to whether homogeneity of these curves with respect to monotone or to open mappins, remain unanswered. However, it follows from the main result obtained in the present paper that (contrary to the mentioned property of the Sierpiński universal plane curve) these two curves are not homogeneous with respect to mappings that are monotone and open simultaneously: the two curves are rational, and each monotone-open mapping defined on a rational continuum must be a homeomorphism. As a corollary it follows that each rational continuum which is homogeneous with respect to monotone-open mappings is a simple closed curve.

2. Preliminaries

All spaces considered in this paper are assumed to be metric, and all mappings are continuous. Given a point c and a number $\varepsilon > 0$, we denote by $B(c, \varepsilon)$ the open ball (in the considered space) with the center c and the radius ε. For a subset A of a space we denote by diam A its diameter. By a *continuum* we mean a compact connected space. A *curve* means a one-dimensional continuum.

The concept of the *order* of a point p in a space X, denoted by $\operatorname{ord}(p, X)$, is used in the sense of Menger-Urysohn (see [14, p. 28]; compare [5, §51, I, p. 274]). Roughly speaking, for a point p of a space X we write $\operatorname{ord}(p, X) = \mathfrak{n}$, provided that $\mathfrak{n}$ is the minimal cardinal number such that there is a local base at p whose elements have boundaries of cardinality $\mathfrak{n}$. If a point p has a local base with finite boundaries, and if the supremum of the cardinalities of the boundaries of elements of the base tends to infinity if their diameters tend to zero, we write $\operatorname{ord}(p, X) = \omega$. A continuum X is said to be *rational* in the sense of the theory of order provided that $\operatorname{ord}(p, X) \leq \aleph_0$ for every point $p \in X$. Thus it follows from the definition of dimension that every rational continuum is a curve, [5, §51, I, p. 275].

A surjective mapping $f : X \to Y$ between continua X and Y is said to be:

- *monotone* provided that for each point $y \in Y$ the set $f^{-1}(y)$ is connected;
- *open* provided that images of open sets under f are open;
- *monotone-open* provided that it is both monotone and open.

3. Results

The main result of this paper is the following theorem.

THEOREM. *Each monotone-open nonconstant mapping defined on a rational continuum is a homeomorphism.*

Proof. Suppose that there is a rational continuum X and a monotone-open nonconstant mapping $f : X \to Y$ which is not a homeomorphism. Then there is a point $q \in Y$ such that $f^{-1}(q)$ is a nondegenerate subcontinuum of X. By openness of f the (monotone) decomposition $\{f^{-1}(y) : y \in Y\}$ of X is continuous (see [14, Chapter 7, Theorem (4.31) and Corollary, p. 130]), and therefore there is an $\varepsilon > 0$ such that for each point $y \in B(q, \varepsilon) \subset Y$ the continuum $f^{-1}(y)$ is nondegenerate. Then $\{f^{-1}(y) : y \in B(q, \varepsilon)\}$ is an uncountable family of pairwise disjoint subcontinua of X, contrary to [5, §51, IV, Theorem 5, p. 285]. The proof is then complete.

Note that rationality of X is an essential assumption in the above theorem, since the projection of the square $X = [0, 1]^2$ onto its side $Y = [0, 1]$ is a monotone-open mapping which is not a homeomorphism. Moreover, even if X is assumed to be a curve, the conclusion does not hold without rationality of the domain continuum X since the Menger universal curve M can be mapped onto any locally connected continuum Y under an open mapping such that point inverses are homeomorphic to M, [15, Theorem 1, p. 497].

Let a space X be given and let $\mathfrak{M}$ be a class of mappings of X onto itself that contains all homeomorphisms, and such that the composition of any two mappings in $\mathfrak{M}$ is also in $\mathfrak{M}$. X is said to be *homogeneous with respect to* $\mathfrak{M}$ provided that, for every two points $p, q \in X$ there is a mapping $f \in \mathfrak{M}$ such that $f(p) = q$. If $\mathfrak{M}$ is the class of homeomorphisms, then X is said to be *homogeneous*.

To conclude a corollary we need a lemma.

LEMMA. *Each rational homogeneous continuum is locally connected.*

Proof. Suppose a continuum X is rational, homogeneous and not locally connected. Thus, by homogeneity, X is locally connected at no of its points. By compactness there is an $\varepsilon > 0$ such that each open subset of X of diameter less than ε is not connected. Let U be any open set with $\operatorname{diam} U < \varepsilon$. Consider the decomposition of $\operatorname{cl} U$ into components. Let $\mathcal{D}$ be the decomposition space, and denote by $\pi : \operatorname{cl} U \to \mathcal{D}$ the projection. Obviously, $\mathcal{D}$ is compact. It is known that it is zero-dimensional (see [5, §46, Va, Theorem 2, p. 150]). We claim that $\mathcal{D}$ contains no isolated point. To this aim take $y \in \mathcal{D}$ and consider two cases. If $\pi^{-1}(y) \subset \operatorname{cl} U \setminus U$, then let $x \in \pi^{-1}(y)$ and let x_n be a sequence of points of U converging to x. Then $\pi(x_n)$ converges to $\pi(x) = y$, so y is not an isolated point of $\mathcal{D}$. If $\pi^{-1}(y) \cap U \neq \emptyset$, take $x \in \pi^{-1}(y) \cap U$. Since there is no connected open subset of U containing x, there is a sequence x_n of points belonging to distinct components of $\operatorname{cl} U$ and converging to x. Then again $\pi(x_n)$ converges to $\pi(x) = y$, so y is not an isolated point of $\mathcal{D}$ as previously. Thus the claim is shown.

Thereby $\mathcal{D}$ is a compact, dense in itself, and zero-dimensional space, so it is homeomorphic to the Cantor ternary set (see e.g. [4, 6.2.A (c), p. 370]). Thus $\operatorname{cl} U$ has uncountably many components. Since each of them intersects the boundary of cl U (see [5, §47, III, Theorem 1, p. 172]), it follows that ord $(p, X) > \aleph_0$, contrary to rationality of X. Thus the proof is complete.

COROLLARY. *Each rational continuum which is homogeneous with respect to monotone-open mappings is a simple closed curve.*

Proof. It follows from the Theorem that any such a continuum X is homogeneous. By the Lemma it is locally connected. Since every homogeneous, locally connected curve is homeomorphic either to a simple closed curve or to the Menger universal curve (see [1, Theorem XIII, p. 14]; compare [6, Theorem 18, p. 96]), and since the Menger universal curve is not a rational curve, the result follows.

Similarly as for the Theorem, the assumption of rationality of the continuum in matter is indespensable for the Corollary, too, even for locally connected plane curves, as it can be seen by homogeneity of the Sierpiński universal plane curve with respect to monotone-open mappings, [8] and [12].

J. J. CHARATONIK
MATHEMATICAL INSTITUTE
UNIVERSITY OF WROCŁAW
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW
POLAND
jjc@hera.math.uni.wroc.pl

INSTITUTO DE MATEMÁTICAS, UNAM
CIRCUITO EXTERIOR
CIUDAD UNIVERSITARIA
04510 MÉXICO, D. F.
MÉXICO
jjc@gauss.matem.unam.mx

W. J. CHARATONIK
MATHEMATICAL INSTITUTE
UNIVERSITY OF WROCŁAW
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW
POLAND
wjcharat@hera.math.uni.wroc.pl

DEPARTAMENTO DE MATEMÁTICAS
FACULTAD DE CIENCIAS, UNAM
CIRCUITO EXTERIOR
CIUDAD UNIVERSITARIA
04510 MÉXICO, D. F.
MÉXICO
wjcharat@lya.fciencias.unam.mx

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