

CONNECTED SUBSETS OF DENDRITES AND SEPARATORS OF THE PLANE

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Connected subsets of dendrites and ones which are locally compact are characterized in the paper. Further, we characterize generalized continua that separate the plane into a finite number of components. Homogeneous families of separators of the plane are also investigated. The results are applied to specify the internal structure of members of a special family of separators of the plane. Several questions are asked.

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The paper consists of four parts. In the first part properties of connected subsets of dendrites are studied. Some characterizations of these spaces are obtained and several necessary and sufficient conditions are found under which such a space is locally compact. The results contained in the first part are then applied in the second one to investigate some special separators of the Euclidean plane. In the third part we study some families of separators, and in the fourth we give a full topological description of a special family of separators of the plane. This family has appeared in a natural way in investigations of multiselections related to a study of some problems in functional analysis, in [15]. Thus all results we have got in the present paper have their roots in analysis, namely in Ricceri's work [15] and in the first named author's conversations with him, to be more precise.

The following standard notation is used in the paper. The set of all positive integers is denoted by \mathbb{N} , and the set of all real numbers by \mathbb{R} . Thus the symbol \mathbb{R}^2 stands for the Euclidean plane. Points of \mathbb{R}^2 are equipped with rectangular coordinates (x, y) . Then

$$D = \{(x, y) \in \mathbb{R}^2: x^2 + y^2 < 1\}$$

is the open unit disc. The symbol $\text{bd } D$ stands for its boundary:

$$\text{bd } D = \text{cl } D \setminus D = \{(x, y) \in \mathbb{R}^2: x^2 + y^2 = 1\}.$$

The usual metric on \mathbb{R}^2 is denoted by d . Given a metric space X , a point $c \in X$ and a positive number r , we put $B(c, r)$ for the open ball in X with center c and radius r .

1. Connected subsets of dendrites

All spaces considered in this part of the paper are assumed to be metric. We start with recalling some definitions of notions used in the paper. A point p of a space X is said to be a *point of order k* in X (writing $\text{ord}_p X = k$, cf. [10, p. 274]) provided there exist arbitrarily small neighborhoods of p in X the boundary of each of which consists of k points, and there are no such neighborhoods whose boundaries are composed of less than k points. A point of order one is called an *end point* of the space. A point of order $k \geq 3$ is called a *ramification point*. Given a space X , the symbols $E(X)$ and $R(X)$ stand for the sets of all end points and of all ramification points of X , respectively.

A *continuum* means a compact connected (metric) space. A locally compact and connected space is called a *generalized continuum* [16, p. 16]. A *dendrite* is defined as a locally connected continuum which contains no simple closed curve. The reader is referred to [10, § 51, VI, pp. 300–303] and to [16, V, 1, pp. 88–89] where important properties of dendrites are presented. An arc contained in a dendrite and such that its end points are end points of the dendrite is called a *maximal arc* of the dendrite.

Spaces X will be considered in the present part of the paper which are connected and can be embedded into a dendrite, i.e., such that there exist a dendrite Y and an embedding $h: X \rightarrow Y$ of X into Y . The internal structure of such spaces is therefore the same as one of a dendrite, with only one change: some maximal arcs are replaced by these arcs without their end points. The details are formulated below.

1.1. Theorem. *The following conditions are equivalent if X is a connected metric space:*

- (i) X is a subset of a dendrite;
- (ii) X can be embedded into a dendrite Y under an embedding $h_1: X \rightarrow Y_1$ in such a way that $h_1(X)$ is a dense subset of Y_1 ;
- (iii) X can be embedded into a dendrite Y_2 under an embedding $h_2: X \rightarrow Y_2$ in such a way that $Y_2 \setminus h_2(X)$ consists of end points of Y_2 .

Proof. Note that implication (iii) \Rightarrow (i) is obvious. To see (i) \Rightarrow (ii) take an embedding $h: X \rightarrow Y$, where Y is a dendrite, and put $Y_1 = \text{cl } h(X)$. Thus Y_1 is a subcontinuum of Y and therefore it is a dendrite [16, V, (1.3), (i), p. 89]. Then $h_1: X \rightarrow Y_1$ defined by $h_1(x) = h(x)$ for $x \in X$ is the needed mapping. Finally observe that the same Y_1 and h_1 can be used for (iii). In fact, if (ii) is assumed, then each point of the difference $Y_1 \setminus h_1(X)$ is an end point of Y_1 since otherwise it is a cut point of Y_1 , and all cut points of Y_1 belong to $h_1(X)$, the latter set being dense in Y_1 . So (iii) follows and the proof is complete. \square

Remark that a connected subset of a dendrite need not be locally compact, and thus it need not be a generalized continuum. To construct such an example take a dendrite Y having a dense set $E(Y)$ of its end points. Then $Y \setminus E(Y)$ has the required property.

1.2. Theorem. *The following conditions are equivalent if X is a connected metric space:*

- (i) X is a locally compact subset of a dendrite;
- (ii) X can be embedded into a dendrite Y_1 under an embedding $h_1: X \rightarrow Y_1$ in such a way that $h_1(X)$ is an open subset of Y_1 ;
- (iii) X can be embedded into a dendrite Y_2 under an embedding $h_2: X \rightarrow Y_2$ in such a way that $h_2(X)$ is a dense open subset of Y_2 ;
- (iv) X can be embedded into a dendrite Y_3 under an embedding $h_3: X \rightarrow Y_3$ in such a way that $Y_3 \setminus h_3(X)$ is a closed subset of Y_3 consisting of its end points;
- (v) X is homeomorphic to a closed and locally connected subset of the Euclidean plane which contains no simple closed curve;
- (vi) each point of X has arbitrarily small neighborhoods being dendrites, and X contains no simple closed curve;
- (vii) X is a locally connected generalized continuum containing no simple closed curve.

Proof. Since a subspace of a Hausdorff space is locally compact if and only if it is homeomorphic to an open subset of its closure [6, Theorem 3.3.9 and Corollary 3.3.11, p. 198], conditions (i) through (iv) are equivalent by Theorem 1.1.

(iv) \Rightarrow (v). Assume (iv), denote $E = Y_3 \setminus h_3(X)$ and consider the quotient space $Y' = Y_3/E$ obtained from Y_3 by shrinking the set E to a point e . Note that Y' is a locally connected continuum with the property that each two simple closed curves in Y' have the point e in common. Thus it contains neither one of the two primitive skew curves of Kuratowski (see [10, § 51, VII, Fig. 11, p. 305]) nor one of the two locally connected skew curves of Claytor (see [4, Figs. 1 and 2, p. 631]). Since a locally connected continuum which is not homeomorphic to a subset of the surface of a sphere necessarily contains one of the four skew curves mentioned above [4, Theorem, p. 631], we conclude that Y' is planable. Thus we can consider Y' as embedded into the plane \mathbb{R}^2 . Let $h: \mathbb{R}^2 \setminus \{e\} \rightarrow \mathbb{R}^2 \setminus \{e\}$ be an inversion of the plane with center e (i.e., the points x and $h(x)$ belong to the same half-line starting from e and $d(e, x) \cdot d(e, h(x)) = 1$). Then $h(Y')$ is a closed subset of the plane homeomorphic to X .

The implications (v) \Rightarrow (vi) \Rightarrow (vii) are obvious.

(vii) \Rightarrow (iv). If X satisfies (vii), then there exists a sequence of dendrites D_i such that for each $i \in \mathbb{N}$ we have $D_i \subset D_{i+1} \setminus E(D_{i+1}) \subset D_{i+1} \subset X$ and that

$$(1) \quad X = \bigcup \{D_i: i \in \mathbb{N}\}.$$

Since each subcontinuum of a dendrite is its monotone retract [8, Theorem, p. 157], for each $i \in \mathbb{N}$ there exists a monotone retraction $f_i: D_{i+1} \rightarrow D_i$. Consider now

the inverse sequence $\{D_i, f_i\}_{i=1}^{\infty}$ and note that the limit $Y_3 = \varprojlim \{D_i, f_i\}$ of this sequence is a dendrite (see [12, Theorem 4, part 3, p. 229]) which obviously contains a homeomorphic copy of the union (1) as a dense subset (compare [1, Theorem I, p. 348]). Further, X is locally compact by [6, Corollary 3.3.10, p. 198], and therefore the copy of X in Y_3 differs from Y_3 by a closed subset $E \subset E(Y_3)$ according to [6, Theorem 3.3.9, p. 198]. The proof is finished. \square

2. Separators

We begin this part of the paper with necessary definitions which are recalled here after [15, p. 223]. A subset C of a space X is called a *separator* of X if there exist two nonempty disjoint open sets A and B in X such that $A \cup B = X \setminus C$. The sets A and B are said to be *associated* to C . In other words a set $C \subset X$ is a separator of X if and only if it is closed and $X \setminus C$ is not connected. If, moreover, there is a connected set $S \subset X$ such that $A \cap S \neq \emptyset \neq B \cap S$, then C is called a *strong separator* of X .

Note the following easy equivalence.

2.1. Proposition. *A T_1 space X is connected if and only if each separator of X is strong.*

Proof. If X is connected, it is enough to put $S = X$ to see one implication. Assume X is not connected. Then there are two nonempty closed proper subsets P and Q of X whose union is X . Take a point $x \in P$. Then the singleton $\{x\}$ is a separator of X , since putting $A = P \setminus \{x\}$ and $B = Q$ we see that both A and B are open, and $X = A \cup \{x\} \cup B$, while there is no connected subset S of X intersecting both A and B . So $\{x\}$ is not a strong separator of X , and the argumentation is complete. \square

2.2. Proposition. *Let a generalized continuum X be such a separator of the plane \mathbb{R}^2 that all components of $\mathbb{R}^2 \setminus X$ are unbounded, and let $n > 1$ be an integer. Then $\mathbb{R}^2 \setminus X$ has n components if and only if X can be embedded into a plane continuum Y which does not separate the plane under an embedding $h: X \rightarrow Y$ such that $Y \setminus h(X)$ consists of n points of Y .*

Proof. First assume there is a continuum Y and an embedding $h: X \rightarrow Y$ satisfying the considered conditions. Take the 2-dimensional sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$ and observe that each component of $\mathbb{R}^2 \setminus X$ is unbounded if and only if $S^2 \setminus X$ is connected. Furthermore, note that if two subsets of the sphere S^2 , say A and B , are homeomorphic, then the numbers of components of their complements $S^2 \setminus A$ and $S^2 \setminus B$ are equal (this is a particular case of a more general result; see [5, Invariance Theorem, p. 73]; see also [9, p. 251]; cf. [10, § 60, VII, Theorem 7, p. 495]). Thereby we see that each component of $\mathbb{R}^2 \setminus h(X)$ is also unbounded. Since $h(X) \subset Y$ is bounded, we infer that $\mathbb{R}^2 \setminus h(X)$ is connected. Thus the n points of $Y \setminus h(X)$ belong

to the boundary of Y , and therefore there exists a simple closed curve $S \subset \mathbb{R}^2$ with $S \cap Y = Y \setminus h(X)$. The bounded component C of $\mathbb{R}^2 \setminus S$ is homeomorphic to the plane, and $C \setminus h(X)$ has exactly n components. One implication is proved.

To show the other one consider a generalized continuum X which is a closed subset of the plane \mathbb{R}^2 and such that $\mathbb{R}^2 \setminus X$ consists of n unbounded components. Let D be the open unit disc. We claim that there is an embedding $g : X \rightarrow \text{cl } D \subset \mathbb{R}^2$ such that $g(X)$ is a closed subset of D and $\text{cl } g(X) \setminus g(X)$ is a proper subset of $\text{bd } D$. So, fix an unbounded component K of $\mathbb{R}^2 \setminus X$ and a closed connected unbounded domain $V \subset K$ whose boundary $\text{bd } V$ is homeomorphic to the real line. Then there is a homeomorphism $f : \mathbb{R}^2 \rightarrow D$ of \mathbb{R}^2 onto D such that $f(V)$ is the closed half-disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \text{ and } y \leq 0\} \subset D$. So $f(X) \subset \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \text{ and } y > 0\}$. Thus putting $g = f|X$ we see that the claim is proved.

Shrinking, if necessary, each nondegenerate component of $\text{cl } g(X) \cap \text{bd } D$ to a point, we can assume without loss of generality that $\text{cl } g(X) \setminus g(X)$ consists of n points at most. We show that there is an embedding $h : X \rightarrow D$ such that $h(X)$ is a closed subset of D and that for each component C of $D \setminus h(X)$ we have

$$(2) \quad (\text{cl } C \setminus C) \cap \text{bd } D \text{ is a nondegenerate subarc of } \text{bd } D.$$

Note that connectedness of $g(X)$ implies connectedness of $(\text{cl } C \setminus C) \cap \text{bd } D$ for each component C of $D \setminus g(X)$. Now we start an inductive procedure. Put $g_0 = g$. Assume we have defined an embedding $g_i : X \rightarrow D$ for some index $i \in \{0, 1, \dots, n\}$ such that there are at least i components C of $D \setminus g_i(X)$ satisfying (2). If there are $i + 1$ components of $D \setminus g_i(X)$ satisfying (2), we put $g_{i+1} = g_i$. Otherwise take a component C' of $D \setminus g_i(X)$ for which $(\text{cl } C' \setminus C') \cap \text{bd } D$ is a singleton $\{b\}$. Take a point $p \in C'$ and join it with b by an arc $A = pb \subset C' \cup \{b\}$. Let

$$f : \text{cl } D \setminus A \rightarrow \text{cl } D \setminus \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \text{ and } y \geq 0\}$$

be a homeomorphism, and put $g_{i+1} = (f|g_i(X))g_i$. Note that $f(C')$ has property (2), i.e., $(\text{cl } f(C') \setminus f(C')) \cap \text{bd } D$ is a nondegenerate subarc of $\text{bd } D$, and that if a component C of $D \setminus g_i(X)$ has property (2), then $f(C)$ has this property, too. Hence there are $i + 1$ components of $D \setminus g_{i+1}(X)$ satisfying (2). Finally put $h = g_n$. Then for each component C of $D \setminus h(X)$ property (2) holds true. There are exactly n such arcs, and therefore there are exactly n points b_1, \dots, b_n (which are end points of these arcs) in the intersection $\text{bd } D \cap \text{cl } h(X)$. The proof is complete. \square

Now let us come back to connected subsets of dendrites discussed in the previous part of the paper. By condition (v) of Theorem 1.2 they can be considered as subsets of the plane, and—under certain conditions—some of them can be separators of the plane. Observe the following obvious fact.

2.3. Proposition. *Let a closed subspace X of the plane be homeomorphic to a connected subset of a dendrite. Then X is a separator of the plane if and only if it contains a homeomorphic copy R of the real line such that for some (equivalently: for each) point p of R both components of $R \setminus \{p\}$ are unbounded.*

The next proposition is a consequence of Proposition 2.2.

2.4. Proposition. *Let a separator X of the plane \mathbb{R}^2 be homeomorphic to a connected subset of a dendrite, and let $n > 1$ be an integer. Then $\mathbb{R}^2 \setminus X$ has n components if and only if X can be embedded into a dendrite Y under an embedding $h: X \rightarrow Y$ such that $Y \setminus h(X)$ consists of n end points of Y .*

2.5. Corollary. *Let a connected separator of the plane be embeddable into a dendrite. Then it separates the plane into two open subsets if and only if it can be densely embedded into a dendrite in such a manner that the remainder consists of exactly two end points of the dendrite.*

2.6. Remark. Note that the conclusion of Proposition 2.4 is no longer true if the number n of components of the complement $\mathbb{R}^2 \setminus X$ is infinite. In fact, the Gehman dendrite G (see [7, the example on p. 42]; cf. [13, p. 422–423], and [14, Fig. 1, p. 203]) can be located in the unit disc in such a way that the set $E(G)$ of its end points (which is homeomorphic to the Cantor ternary set) is contained in the boundary of the disc, while all other points of the dendrite lie in the interior of the disc. Then $G \setminus E(G)$ is a local dendrite which separates the interior of the disc into countably many domains (i.e., open connected sets), while $G \setminus E(G)$ cannot be embedded into a dendrite having countably many end points only.

3. Families of separators

We start with the following result.

3.1. Theorem. *Let \mathcal{F} denote an arbitrary family of pairwise disjoint connected unbounded separators of the plane. For each element C of \mathcal{F} there exists an open set V containing C such that C is the only element of \mathcal{F} which is contained in V .*

Proof. Fix an unbounded element C of \mathcal{F} and let a subfamily \mathcal{E} of $\mathcal{F} \setminus \{C\}$ consist of such separators C' for which all elements of $\mathcal{F} \setminus \{C, C'\}$ are contained in just one component of $\mathbb{R}^2 \setminus C'$. It can be verified that \mathcal{E} is at most countable. Denote by F the union of all members of \mathcal{F} distinct from C . Consider, for $n \in \mathbb{N}$, the pairs (a_n, b_n) of points of F satisfying the following conditions:

- (3) both a_n and b_n belong to the same component of $\mathbb{R}^2 \setminus C$;
- (4) for each two points $a, b \in F$ lying in the same component of $\mathbb{R}^2 \setminus C$ and for every $\varepsilon > 0$ there exists an index $n \in \mathbb{N}$ such that $d(a_n, a) < \varepsilon$ and $d(b_n, b) < \varepsilon$;
- (5) for each $C' \in \mathcal{E}$ there exists $n \in \mathbb{N}$ such that $a_n \in C'$.

Now we define by induction a special sequence of pairs of points (c_n, d_n) and of arcs L_n joining these points. Put $c_1 = a_1$ and $d_1 = b_1$, and let $L_1 \subset \mathbb{R}^2 \setminus C$ be an arc

joining c_1 and d_1 . Fix a point $x_0 \in C$, and put $r_1 = \sup\{d(x_0, x) : x \in L_1\}$. Since all elements of \mathcal{F} are unbounded, one can choose some points c_2 and d_2 such that

- (6) there are two members of \mathcal{F} such that one of them contains a_2 and c_2 and the other contains b_2 and d_2 ; and
- (7) there exists an arc L_2 with end points c_2 and d_2 lying out of the union $C \cup B(x_0, r_1 + 1)$.

Put $r_2 = \sup\{d(x_0, x) : x \in L_1 \cup L_2\}$. Assume that the pairs (c_n, d_n) and the arcs L_n with end points c_n, d_n are defined for some $n \in \mathbb{N}$. Put $r_n = \sup\{d(x_0, x) : x \in L_1 \cup \dots \cup L_n\}$. Since elements of \mathcal{F} are unbounded, there exist points c_{n+1} and d_{n+1} such that there are two members of \mathcal{F} one of which contains a_{n+1} and c_{n+1} and the other one contains b_{n+1} and d_{n+1} , and that there exists an arc L_{n+1} joining c_{n+1} and d_{n+1} and lying out of the union $C \cup B(x_0, r_n + 1)$. It can be seen from the construction that the union $\bigcup\{L_n : n \in \mathbb{N}\}$ is a closed subset of the plane. Thus the set $V = \mathbb{R}^2 \setminus \bigcup\{L_n : n \in \mathbb{N}\}$ is open. Obviously it contains the fixed separator $C \in \mathcal{F}$. We show that V has the needed property. So, suppose on the contrary that there is a member C_1 of $\mathcal{F} \setminus \{C\}$ with $C_1 \subset V$. Then C_1 is not in \mathcal{E} by condition (5) above. Therefore by condition (4) there exists an index $n \in \mathbb{N}$ such that the points a_n and b_n lie in distinct components of $\mathbb{R}^2 \setminus C_1$. Thus the points c_n and d_n also lie in distinct components of $\mathbb{R}^2 \setminus C_1$, and so we have $L_n \cap C_1 \neq \emptyset$, contrary to the definition of V . The proof is then complete. \square

In [15, pp. 223 and 224], Ricceri has introduced the concept of a homogeneous family of separators of a space. Namely a family $\{C_i : i \in I\}$ of separators of a space X is said to be *homogeneous* provided there exist two families $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ of subsets of X such that for each $i \in I$ the sets A_i and B_i are associated to C_i , and for every open connected set $U \subset X$ intersecting both A_i and B_i there is an open set V containing C_i and such that if $C_j \subset V$ for some $j \in I$, then

$$(8) \quad A_j \cap U \neq \emptyset \neq B_j \cap U.$$

Using the above concept, the following corollary is an immediate consequence of Theorem 3.1.

3.2. Corollary. *Every family of pairwise disjoint connected and unbounded separators of the plane is homogeneous.*

3.3. Remark. Both in Theorem 3.1 and in Corollary 3.2 we can assume that members of \mathcal{F} are unbounded except of a subfamily of \mathcal{F} having the closed union. In fact, the constructed open set V may be chosen out of the (closed) union of the considered subfamily.

3.4. Remarks. Now we will discuss necessity of assumptions made in Theorem 3.1 and Corollary 3.2.

(1) The members of \mathcal{F} have to be unbounded. Indeed, put $L = \{(x, y) \in \mathbb{R}^2: y = 0\}$ and, for an arbitrary positive integer n , let C_n denote the circle with center $(0, 2^{-n})$ and radius 2^{-n-2} . Then $\mathcal{F} = \{L\} \cup \{C_n: n \in \mathbb{N}\}$ is not a homogeneous family of separators of \mathbb{R}^2 (and therefore the conclusion of Theorem 3.1 does not hold).

(2) The members of \mathcal{F} have to be pairwise disjoint. To see this, keep the previous denotation. Let I_n be the straight line segment joining the origin $(0, 0)$ with the lowest point $(0, 3 \cdot 2^{-n-2})$ of the circle C_n . For each $n \in \mathbb{N}$ put $D_n = L \cup C_n \cup I_n$. Then $\mathcal{F} = \{L\} \cup \{D_n: n \in \mathbb{N}\}$ is again a nonhomogeneous family of separators of \mathbb{R}^2 .

3.5. Remark. In [15, p. 224], Ricceri has asked the following question. Let a space X be locally connected and metrizable, and let $\{C_i: i \in I\}$ be a homogeneous family of separators of X which, moreover, is a decomposition of X (i.e., X is the union of all members of the family) and has the continuum power. Then, is there some continuous function $f: X \rightarrow [0, 1]$ for which the equality

$$\{C_i: i \in I\} = \{f^{-1}(t): t \in]0, 1[\cap f(X)\}$$

holds? The above question has been answered in the negative in [3] even in two ways: for acyclic and for cyclic locally connected curves X by showing corresponding examples. The results contained in Theorem 3.1 and Corollary 3.2 can be considered as a more complete answer to the Ricceri question. However, going along Ricceri's ideas, his question can be modified as follows.

3.6. Problem. Let a family \mathcal{F} of separators of a topological space X be given. What are sufficient and/or necessary conditions for \mathcal{F} under which the quotient space X/\mathcal{F} is homeomorphic to the real line?

In the next section we modify the concept of homogeneity of a family of separators to obtain some conditions mentioned in the above problem, under an additional (but the most interesting) assumption that the space X under consideration is just the Euclidean plane.

4. Decompositions

By a *decomposition* of a space X we understand a family of pairwise disjoint closed subsets of X whose union is X . A subset A of X is called a *retract* of X (see [2, p. 10]) if there is a continuous mapping f of X onto A (called a *retraction*) such that $f(x) = x$ for $x \in A$.

After reading the main results of [3], Ricceri has asked (in a conversation with the first named author) if the function f considered here in Remark 3.5 does exist provided that we take the plane \mathbb{R}^2 as the space X and we assume that each member of the considered homogeneous family of pairwise disjoint separators of the plane is a nowhere dense retract of \mathbb{R}^2 . Since connectedness and local connectedness are

invariants under retractions [2, p. 18; (8.2), p. 19; (2.1), p. 10], each member of the family under consideration is a closed, connected and locally connected subset of the plane. Furthermore, it is known that if a retract C of the Euclidean n -dimensional space \mathbb{R}^n , where $n > 1$, is bounded, then its complement $\mathbb{R}^n \setminus C$ is connected. Since each member of the considered family is a separator of the plane, we conclude that the family consists of unbounded subsets of \mathbb{R}^2 . But then we infer from our Corollary 3.2 that homogeneity of the family does not suffice to attain the existence of the needed function f . As it was already said above, to get the conclusion we modify this concept in such a way that the result can be obtained, even under some weaker assumptions on the members of the considered family.

A family $\{C_i: i \in I\}$ of separators of a space X is said to be *strongly homogeneous* provided there exist two families $\{A_i: i \in I\}$ and $\{B_i: i \in I\}$ of subsets of X such that for each $i \in I$ the sets A_i and B_i are associated to C_i , and for every open connected set $U \subset X$ intersecting both A_i and B_i there is an open set V containing C_i and such that if $C_j \cap V \neq \emptyset$ for some $j \in I$, then

$$(8) \quad A_j \cap U \neq \emptyset \neq B_j \cap U.$$

Let $\Delta = \{C_i: i \in I\}$ be a strongly homogeneous family of nowhere dense, connected and locally connected pairwise disjoint (strong) separators of \mathbb{R}^2 , with $\bigcup \{C_i: i \in I\} = \mathbb{R}^2$. Below a sequence of properties of the decomposition Δ of the plane is presented which leads to a characterization of an internal structure of each member of Δ . Our first statement is an immediate consequence of the Baire category theorem, since each member C_i of Δ is nowhere dense by assumption.

4.1. The family Δ is uncountable.

Since each separator of a space is a closed subset of the space by the definition, then:

4.2. Each member of Δ is a closed, connected and locally connected subset of \mathbb{R}^2 .

The next observation is the following.

4.3. Each member of Δ is an unbounded subset of \mathbb{R}^2 .

Indeed, assume there exists a bounded member of the family Δ . Consider the partial order $<$ on Δ defined by $A < B$ provided that A is contained in a bounded component of $\mathbb{R}^2 \setminus B$. Then, by the Cantor theorem on the intersection of compact sets, each linearly ordered subfamily of Δ has a lower bound. However, the family Δ has no minimal element, contrary to the Kuratowski-Zorn lemma.

4.4. No member of Δ contains a simple closed curve.

To see this, suppose the contrary. Let K be a simple closed curve contained in a member C_i of Δ . Then by the Jordan curve theorem (see [10, § 61, II, Theorem

1, p. 510]) K is the boundary of the bounded component B of $\mathbb{R}^2 \setminus K$. Thus $K \cup B$ is a closed disc, and therefore it cannot be a subset of C_i which is nowhere dense. Hence there is a point b in $B \setminus C_i$. Since the family Δ is a decomposition of \mathbb{R}^2 , there is a member C_j of Δ to which the point b belongs. Since C_j is connected by 4.2 above, and since K disconnects \mathbb{R}^2 into two components (again by the Jordan theorem) one of which is B , the condition $b \in B \cap C_j$ implies $C_j \subset B$. But then C_j is bounded, contrary to 4.3.

Now, observations 4.2 and 4.4 and the implication (vi) \Rightarrow (i) of Theorem 1.2 enable us to say that each member of Δ is homeomorphic to a locally compact subset of a dendrite. By 4.3 it cannot be compact. Since every connected subset of a dendrite is an absolute retract [10, § 54, VII, Corollary 7, p. 378], we have our next assertion.

4.5. *Each member of Δ is homeomorphic to a noncompact, locally compact subset of a dendrite, and therefore it is a retract of the plane.*

4.6. *Only countably many members of the family Δ can contain ramification points. All other ones are then homeomorphic to the real line.*

In fact, since every ramification point in a dendrite is a common point of at least three arcs which are disjoint out of this point (by the Menger n -arc theorem, see [10, § 51, I, 8, p. 277]), each member of Δ having a ramification point contains a simple triod. Since members of Δ are pairwise disjoint, the triods taken in disjoint members of Δ are disjoint, too. However, by the Moore triodic theorem [11, Theorem, p. 262] each uncountable collection of triods lying in the plane contains an uncountable subcollection every two elements of which have a point in common. Therefore each family of pairwise disjoint triods in the plane is at most countable, and so 4.6 is proved.

Remark that each dendrite has at most countably many ramification points [10, § 51, VI, Theorem 7, p. 302]. Hence it follows from 4.6 that the union of the sets of all ramification points of members of Δ , taken over all members of the family Δ , is still a countable set.

Our next assertion says that, in fact, for each member C_i of the family Δ the associated sets A_i and B_i can be uniquely determined. This is stated in a more precise way as follows.

4.7. *For each member C_i of the family Δ its complement $\mathbb{R}^2 \setminus C_i$ has exactly two components.*

Assume on the contrary that there is a member C_i of Δ such that $\mathbb{R}^2 \setminus C_i$ has at least three components. Denote by K one of them. Since C_i is homeomorphic to a connected subset of a dendrite, there exists a point $p \in C_i \setminus \text{cl } K$. Let U be an open neighborhood of p disjoint with $\text{cl } K$, and V be an open subset of \mathbb{R}^2 containing C_i and satisfying the conditions mentioned in the definition of strong homogeneity of

Δ . Further, let C_j be any member of Δ which intersects $K \cap V$. Then $C_j \subset K$, and therefore either A_j or B_j does not intersect U , contrary to the definition of strong homogeneity of Δ . The proof is finished.

Assertions 4.5, 4.7 and Corollary 2.5 lead to the following statement.

4.8. *Each member of the family Δ can be densely embedded into a dendrite in such a way that the remainder consists of exactly two end points of the dendrite.*

Therefore the following result has been proved.

4.9. Theorem. *Let there be given a strongly homogeneous family Δ of nowhere dense, pairwise disjoint separators of the plane \mathbb{R}^2 , whose union is \mathbb{R}^2 . Then the following conditions are equivalent.*

- (i) *Each member of Δ is a connected and locally connected subset of the plane.*
- (ii) *Each member of Δ is a retract of the plane.*
- (iii) *All members of Δ except at most countably many of them are homeomorphic to the real line. All other members of Δ are homeomorphic to noncompact, locally compact subsets of dendrites. Each of them separates \mathbb{R}^2 into exactly two components, and thus can be embedded into a dendrite in such a manner that the remainder consists of two end points of the dendrite.*

Let us remark that strong homogeneity of the family Δ was used in the proof of statement 4.7 only. Thus the following problem seems to be natural.

4.10. Problem. Let a decomposition of the plane \mathbb{R}^2 be a strongly homogeneous family \mathcal{F} of its separators. Find necessary and sufficient conditions under which for each member M of \mathcal{F} its complement $\mathbb{R}^2 \setminus M$ has exactly two components.

Note that, according to Theorem 4 of [10, § 46, VIII, p. 159], the property surely holds for all except at most countably many members M of the considered family.

4.11. Remark. We show now it is not enough to assume that all separators are nowhere dense only. In other words, elements of the family Δ considered above have to be locally connected to obtain 4.7. To this aim consider the common boundary B of three unbounded domains R_1, R_2, R_3 in the plane \mathbb{R}^2 . Note that B is not locally connected. Each of these domains is homeomorphic to \mathbb{R}^2 . Let $h_i: R_i \rightarrow \mathbb{R}^2$ for $i \in \{1, 2, 3\}$ be homeomorphisms. Put $L_x = \{(x, y): y \in \mathbb{R}\}$ and let

$$\mathcal{F} = \{B\} \cup \{h_i(L_x): x \in \mathbb{R} \text{ and } i \in \{1, 2, 3\}\}.$$

One can verify that \mathcal{F} is a strongly homogeneous family of separators of \mathbb{R}^2 , and of course $\mathbb{R}^2 \setminus B$ has three components. Further, the quotient space $\mathbb{R}^2 / \mathcal{F}$ is homeomorphic to the simple triod without its end points.

Now we prove that the decomposition space \mathbb{R}^2/Δ is homeomorphic to the real line. We show even a stronger result.

4.12. Theorem. *Let a decomposition \mathcal{F} of the plane \mathbb{R}^2 into its separators be such that the complement of each member of \mathcal{F} has exactly two components. Then \mathbb{R}^2/\mathcal{F} is homeomorphic to the real line.*

Proof. The space \mathbb{R}^2/\mathcal{F} is separable and connected as a continuous image of the plane. To see it is homeomorphic to \mathbb{R} it is enough to show that it is linearly ordered and has neither the lowest nor the greatest element. To define the order $<$ on \mathcal{F} choose an element $C \in \mathcal{F}$, and denote the components of $\mathbb{R}^2 \setminus C$ by A and B . Consider two distinct separators F and F' of \mathcal{F} . We say $F < F'$ if: (1) $F \subset A \cup C$ and $F' \subset B$, or (2) $F \cup F' \subset A \cup C$ and $C \cup F'$ is contained in a component of $\mathbb{R}^2 \setminus F$, or (3) $F \cup F' \subset B \cup C$ and $C \cup F$ is contained in a component of $\mathbb{R}^2 \setminus F'$. One can verify that $<$ is a linear order on \mathcal{F} , there are neither the lowest nor the greatest element in this ordering, and the quotient topology on \mathbb{R}^2/\mathcal{F} does agree with the one defined by the order. The proof is finished. \square

4.13. Corollary. *The decomposition space \mathbb{R}^2/Δ is homeomorphic to the real line, i.e., there is a continuous function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ (which in general need not be either closed or open) such that*

$$\Delta = \{f^{-1}(t): t \in \mathbb{R}\}.$$

In connection with the above theorem, Ricceri has asked, in a conversation with the first named author, if it is possible to specify the function f considered in Corollary 4.12 in a more precise way. Before we formulate his question, recall two definitions. Let d denote the Euclidean metric on \mathbb{R}^2 . A function $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ is said to be *Lipschitzian* provided that there is a positive real number k (called the *Lipschitz constant*) such that, for each two points (x', y') and (x'', y'') of the plane \mathbb{R}^2 the inequality

$$|g(x', y') - g(x'', y'')| \leq k \cdot d((x', y'), (x'', y''))$$

holds true. The *Hausdorff distance* d_H between two closed subsets A, B of the plane \mathbb{R}^2 is defined by the formula

$$d_H(A, B) = \max\{\sup\{d(a, B): a \in A\}, \sup\{d(b, A): b \in B\}\}.$$

Now the Ricceri question runs as follows.

4.14. Question (Ricceri). Let a decomposition Δ of the plane \mathbb{R}^2 be given which satisfies all the hypotheses of Theorem 4.9 and moreover such that for each two members C_i, C_j of Δ their Hausdorff distance $d_H(C_i, C_j)$ is finite. Under what additional assumptions on Δ are there two real numbers a and b with $a^2 + b^2 > 0$,

and a Lipschitzian function $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ with a Lipschitz constant $k < \sqrt{a^2 + b^2}$ such that if we put

$$f(x, y) = ax + by + g(x, y) \quad \text{for } (x, y) \in \mathbb{R}^2,$$

then we have $\Delta = \{f^{-1}(t): t \in \mathbb{R}\}$?

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