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Smooth dendroids as inverse limits of dendrites

by

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Abstract. It is proved that a dendroid is smooth if and only if it can be represented as the inverse limit of an inverse sequence of finite dendrites with bonding mappings which are monotone relative to points forming a thread. As a consequence another proof of the existence of a universal smooth dendroid [4] is obtained.

§ 1. **Preliminaria.** All spaces considered in this paper are assumed to be metric and all mappings are continuous. A *dendroid* means a hereditarily unicoherent and arcwise connected continuum. If, moreover, it is locally connected, it is called a *dendrite*. By a *ramification point* of a dendroid X we understand a point which is the centre of a simple triod contained in X. A dendroid having at most one ramification point t is called a *fan*, and t is called its *top*. A fan with at most n endpoints is called an *n-fan*. A dendroid X is said to be *smooth at a point* $p \in X$ provided that for each sequence of points $a_n \in X$ which is convergent to a point $a \in X$ the sequence of arcs pa_n converges to the arc pa. A mapping $f\colon X \to Y$ of a continuum X onto Y is said to be *monotone relative to a point* $p \in X$ if for each continuum Q in Y such that $f(p) \in Q$ the set $f^{-1}(Q)$ is connected (see [6], p. 720).

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§ 2. **The main result and corollaries.** The following result is a particular case (for dendrites) of Corollary 4 of [1].

THEOREM A. *Let an inverse sequence* $\{X^i, f^i\}$ *be given of dendrites* X^i *containing points* p^i *such that* 1^0 $f^i(p^{i+1}) = p^i$ *and* 2^0 *the bonding mappings* $f^i\colon X^{i+1} \to X^i$ *are monotone relative to points* p^{i+1}. *Then the inverse limit* $X = \varprojlim\{X^i, f^i\}$ *is a dendroid which is smooth at the thread* $p = \{p^i\}$.

The aim of this paper is to prove the inverse theorem, so that the characterization can be obtained of smooth dendroids as inverse limits of finite dendrites with bonding mappings which are monotone relative to some points forming a thread of the inverse sequence. Namely we shall prove the following

MAIN THEOREM. *Let a dendroid X be given which is smooth at a point $p \in X$. Then for each $i \in \{1, 2, \ldots\}$ there exist finite dendrites X^i, mappings $f^i\colon X^{i+1} \to X^i$ and points $p^i \in X^i$ such that* 1^0 $f^i(p^{i+1}) = p^i$ *and* 2^0 *the mappings f^i are monotone relative to p^{i+1} and the inverse limit $\underleftarrow{\mathrm{Lim}}\{X^i, f^i\}$ is homeomorphic to X in such a way that the thread $\{p^i\}$ corresponds to the point p.*

Before proving the theorem we pose some problems and give corollaries.

PROBLEM. What continua X can be obtained as inverse limits of locally connected continua X^i with bonding mappings f^i satisfying conditions 1^0 and 2^0 of Theorem A?

In a proposition below we give an answer to this question for the class of continua which are hereditarily unicoherent at a point. Recall that a continuum X is said to be *hereditarily unicoherent at a point $p \in X$* if the intersection of any two subcontinua of X each of which contains p is connected.

PROPOSITION 1. *Let an inverse sequence $\{X^i, f^i\}$ be given of locally connected continua X^i containing points p^i such that conditions* 1^0 *and* 2^0 *of Theorem* A *hold. If the inverse limit $X = \underleftarrow{\mathrm{Lim}}\{X^i, f^i\}$ is hereditarily unicoherent at the thread $p = \{p^i\}$, then each X^i is a dendrite and hence X is a dendroid which is smooth at p.*

In fact, by Corollary 1 of [2] each natural projection from X onto X^i is monotone relative to p. Hence, by Theorem 2.5 of [6], p. 721, each X^i is hereditarily unicoherent at p^i. So X^i is a dendrite by Theorem 2.2 of [3], p. 63.

Now we are interested in the universal smooth dendroid. Its existence has been proved in [4]. We show that the standard methods of McCord [7] applied to the class of pointed finite dendrites with mappings monotone relative to distinguished points, together with the Main Theorem, give another proof of the existence of a universal smooth dendroid. For this purpose we need some auxiliary concepts.

A pair (X, x) where $x \in X$ is called a pointed space. Let $\mathscr{K}$ be a class of mappings of pointed spaces which is closed with respect to taking compositions. The class $\mathscr{P}$ of pointed polyhedra is called *$\mathscr{K}$-amalgamable* if for each finite sequence $(P_1, p_1), (P_2, p_2), \ldots, (P_n, p_n)$ of members of $\mathscr{P}$ and mappings $f_i\colon (P_i, p_i) \to (Q, q)$ where $(Q, q) \in \mathscr{P}$ and each $f_i \in \mathscr{K}$ there exist a member (P, p) of $\mathscr{P}$ with embeddings $g_i\colon (P_i, p_i) \to (P, p)$ and a mapping $f \in \mathscr{K}$ of (P, p) onto (Q, q) such that $f_i = f g_i$ for each $i \in \{1, 2, \ldots, n\}$.

Let $\mathscr{P}$ be a class of pointed polyhedra. We say that a pointed continuum (X, x) is *$(\mathscr{P}, \mathscr{K})$-like* if there is an inverse sequence of members of $\mathscr{P}$ with bonding mappings belonging to $\mathscr{K}$ such that (X, x) is the inverse limit of that sequence.

Using exactly the same arguments as McCord uses in his proof of Theorem 1 of [7], Part 3, p. 72–77 and considering the concepts introduced above, we get

PROPOSITION 2. *If a class $\mathscr{P}$ of pointed polyhedra is $\mathscr{K}$-amalgamable, then there exists a universal $(\mathscr{P}, \mathscr{K})$-like continuum.*

Denote by $\mathscr{D}$ the class of pointed finite dendrites and by $\mathscr{M}$ the class of mappings which are monotone relative to distinguished points. Now we are able to reformulate the Main Theorem and Theorem A as follows:

COROLLARY 1. *A pair (X, p) is a pointed dendroid which is smooth at p if and only if (X, p) is $(\mathscr{D}, \mathscr{M})$-like.*

PROPOSITION 3. *The class $\mathscr{D}$ is $\mathscr{M}$-amalgamable.*

Proof. Let $(P_1, p_1), \ldots, (P_n, p_n)$ be pointed finite dendrites and let $f_i\colon (P_i, p_i) \to (Q, q)$ be monotone relative to p_i with $(Q, q) \in \mathscr{D}$. Let P be the one-point union of P_i with points $p_1, p_2, \ldots, p_n$ identified to a point $p \in P$. Then (P, p) is a pointed finite dendrite. We consider g_i as a natural embedding of (P_i, p_i) into (P, p) and define $f\colon (P, p) \to (Q, q)$ by $f|P_i = f_i$, or — more exactly — $f(x) = f_i(g_i^{-1}(x))$ for $x \in g_i(P_i)$. One can observe (simply by definitions) that f is monotone relative to p. So all conditions of the definition are satisfied.

Corollary 1 and Propositions 2 and 3 lead to

COROLLARY 2 ([4], Theorem 3.1, p. 992). *There exists a universal smooth dendroid.*

§ 3. **Proof of the Main Theorem.** The following result of Maćkowiak will be used in the sequel.

THEOREM B ([6], Corollary 2.10, p. 722). *Let a continuous mapping f map a dendroid X onto a dendroid Y, and let $p \in X$. Then f is monotone relative to p if and only if $f|px$ is monotone for each $x \in X$.*

For each natural number i let F^i be the cone over the set $A^i = \{0, 1\}^i$ and let F be the cone over the Cantor set $C = \{0, 1\}^\infty$; i.e., $F^i = A^i \times [0, 1]/A^i \times \{0\}$ and $F = C \times [0, 1]/C \times \{0\}$ are the 2^i-fan and the Cantor fan respectively. Denote by t^i the top of the fan F^i and by t the top of the fan F. The projections $C \to A^i$ and $A^{i+1} \to A^i$ induce maps $p^i\colon F \to F^i$ and $u^i\colon F^{i+1} \to F^i$.

We shall employ the following result of Grispolakis and Tymchatyn.

THEOREM C ([5], Theorem 2.3, p. 132). *Each smooth dendroid X can be embedded into a smooth dendroid D_X such that there exists a mapping $g\colon F \to D_X$ satisfying the conditions:*

1^0 *if* $g(x_1, y_1) = g(x_2, y_2)$, *then* $y_1 = y_2$ *for each* $(x_1, y_1), (x_2, y_2) \in F$;

2^0 *if* $g(x_1, y_1) = g(x_2, y_1)$ *and* $0 < y < y_1$, *then* $g(x_1, y) = g(x_2, y)$;

3^0 *for each* $y \in [0, 1]$ *the set* $g(C \times \{y\})$ *is zero-dimensional*;

4^0 $g(t) \in X \subset D_X$.

The main step in the proof of the Main Theorem is

PROPOSITION 4. *Suppose D_X is a smooth dendroid, $X \subset D_X$ and a map $g\colon F \to D_X$ satisfies 1^0–4^0 of Theorem C. Then for each $i \in \{1, 2, \ldots\}$ there are finite dendrites D^i and maps $g^i\colon F^i \to D^i$, $v^i\colon D^{i+1} \to D^i$ and $q^i\colon D_X \to D^i$ such that*

(a) *D^i is a dendrite with at most 2^i end-points;*

(b) *the diagram*

$$\begin{tikzcd}
F^i \arrow[d, "g^i"'] & F^{i+1} \arrow[l, "u^i"'] \arrow[d, "g^{i+1}"] & F \arrow[l, "p^{i+1}"'] \arrow[d, "g"] \\
D^i & D^{i+1} \arrow[l, "v^i"] & D_X \arrow[l, "q^{i+1}"]
\end{tikzcd}$$

commutes and all mappings are monotone relative to points $t^i \in F^i$, $t^{i+1} \in F^{i+1}$, $t \in F$, $g^{i+1}(t^{i+1}) \in D^{i+1}$ *and* $g(t) \in D_X$ *respectively. Moreover,*

(c) *if* $d_1, d_2 \in D_X$ *and* $d_1 \neq d_2$, *then there exists an index* i *with* $q^i(d_1) \neq q^i(d_2)$.

Proof. For each $i \in \{1, 2, \ldots\}$ consider the family $\mathscr{R}^i$ of all relations R satisfying four conditions:

(1) R is a closed subset of $F^i \times F^i$;

(2) if $(x_1, y_1)R(x_2, y_2)$, then $y_1 = y_2$, where $x_1, x_2 \in A^i$ and $y_1, y_2 \in [0, 1]$;

(3) if $(x_1, y_1)R(x_2, y_1)$ and $0 < y < y_1$, then $(x_1, y)R(x_2, y)$;

(4) for each two points $f_1, f_2 \in F$, if $g(f_1) = g(f_2)$, then $p^i(f_1)Rp^i(f_2)$.

One can verify in a routine way than this family is multiplicative. To see that it is non-empty define $R \subset F^i \times F^i$ by $(x_1, y_1)R(x_2, y_2)$ if and only if $y_1 = y_2$ and note $R \in \mathscr{R}^i$. Put $R^i = \bigcap \mathscr{R}^i$ and note $R^i \in \mathscr{R}^i$. Define $D^i = F^i/R^i$ and $g^i\colon F^i \to D^i$ as the identification map.

We show (a). The following properties of the mapping g^i are consequences of (2) and (3):

(5) if $g^i(x_1, y_1) = g^i(x_2, y_2)$, then $y_1 = y_2$ and

(6) if $g^i(x_1, y_1) = g^i(x_2, y_1)$ and $0 < y < y_1$, then $g^i(x_1, y) = g^i(x_2, y)$.

By a straightforward induction on n it follows that a map defined on an n-fan satisfying (5) and (6) has a dendrite with at most n end-points as its image. So (a) is established.

Now we define $v^i\colon D^{i+1} \to D^i$. Take a point $d \in D^{i+1}$ and let $f \in F^{i+1}$ satisfy $g^{i+1}(f) = d$. Put $v^i(d) = g^i(u^i(f))$. To see that the definition is correct consider the relation R defined on F^{i+1} by f_1Rf_2 if and only if $g^i(u^i(f_1)) = g^i(u^i(f_2))$ and note that R satisfies (1)–(4) for F^{i+1} and hence $R^{i+1} \subset R$. This means that $g^{i+1}(f_1) = g^{i+1}(f_2)$ implies $g^i(u^i(f_1)) = g^i(u^i(f_2))$ and we are done.

Similarly define $q^i\colon D_X \to D^i$ by $q^i = g^i p^i (g)^{-1}$. This definition is correct by (4). The commutativity of the diagram follows directly from the definitions of v^i and q^i.

Observe that the mappings p^i, u^i, g^i and g are monotone relative to respective points by Theorem B. To see that so is v^i, consider a continuum $Q \subset D^i$ with $g^i(t^i) \in Q$ and observe that $(v^i)^{-1}(Q) = g^{i+1}(u^i)^{-1}(g^i)^{-1}(Q)$ is a continuum by monotoneity relative to the respective points of the mappings g^i and u^i. A similar argument implies q^i is monotone relative to $g(t)$, and so (b) is established.

It remains to show (c), i.e., that if $f_1, f_2 \in F$ with $g(f_1) \neq g(f_2)$ then $g^i(p^i(f_1)) \neq g^i(p^i(f_2))$ for some index i. To this end write $f_j = (x_j, y_j)$, where $x_j \in C$ and

$y_j \in [0,1]$ for $j \in \{1,2\}$. If $y_1 \neq y_2$ then by (5) $g^i(p^i(f_1)) \neq g^i(p^i(f_2))$, and so assume $y_1 = y_2 = y \in (0,1]$. By condition 3^0 of Theorem C the set $g(C\times\{y\})$ is zero-dimensional, whence there are two closed and open sets U_1 and $U_2 = g(C\times\{y\})\setminus U_1$ containing the points $g(f_1)$ and $g(f_2)$ respectively. Write $g^{-1}(U_j) = C_j\times\{y\}$ for $j\in\{1,2\}$. So C_1 and C_2 are disjoint, closed and open subsets of C satisfying $C_1 \cup C_2 = C$ and $x_1 \in C_1$, $x_2 \in C_2$.

Since $g(x_1, y) \neq g(x_2, y)$ and since g is continuous, there exists a positive number $z<y$ with $g(x', z) \neq g(x'', z)$ for each $x' \in C_1$ and $x'' \in C_2$. Note that condition 2^0 of Theorem C implies

(7) $g(x', z') \neq g(x'', z')$ for all $z' \in [z, 1]$, $x' \in C_1$ and $x'' \in C_2$.

Observe that C is the inverse limit of the sets A^i with the projections $A^{i+1} \to A^i$ as bonding mappings. Denoting by $r^i\colon C \to A^i$ the projection map, we see that there exists an index i such that the sets $r^i(C_1)$ and $r^i(C_2)$ are non-empty, disjoint subsets of A^i with $r^i(C_1) \cup r^i(C_2) = A^i$.

Define a relation R on F^i, putting $(a_1, b_1)R(a_2, b_2)$ if and only if $b_1 = b_2 = b \in [0,1]$ and

$b \leqslant z$, or

$b > z$ and $a_1, a_2 \in r^i(C_1)$, or

$b > z$ and $a_1, a_2 \in r^i(C_2)$.

Observe that the relation R satisfies conditions (1)–(4) ((4) is a consequence of (7)). So $R^i \subset R$. Note that

$$(p^i(f_1), p^i(f_2)) \in F^i\times F^i\setminus R \subset F^i\times F^i\setminus R^i$$

hence $g^i(p^i(f_1)) \neq g^i(p^i(f_2))$, which establishes (c) and finishes the proof of Proposition 4.

Proof of the Main Theorem. Let X be a subset of D_X as in Theorem C. In the notation of Proposition 4 let $X^i = q^i(X)$ and $f^i = v^i|X^{i+1}$. Then X^i is a subcontinuum of D^i, and so it is a dendrite with at most 2^i end-points; further, f^i is monotone relative to $g^i(t^i)$ by Proposition 3 of [1]. Define $h\colon X \to \underleftarrow{\operatorname{Lim}}\{X^i, f^i\}$ putting $h(x) = \{p^i(x)\}$ for $x \in X$. It follows from (c) that h is a homeomorphism, and so the proof is complete.

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