INVERSE LIMITS OF CONTINUOUS HAVING TRIVIAL SHAPE

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Abstract. We show that trivial shape is preserved under inverse limits of continua with multivalued bonding functions provided the images of points have trivial shape. Some examples are provided.

1. Introduction

Ingram and Mahavier [3] began the investigation of inverse limits of upper semi-continuous set valued functions. In this paper we continue this investigation by considering the inverse limit of continua having trivial shape with bonding functions being upper semi-continuous continuum valued functions.

Only a few topological properties are known to be preserved by inverse limits with upper semi-continuous set valued bonding functions. Two examples known to the authors are, Ingram and Mahavier in [3] have given conditions under which connectedness and compactness are preserved and Nall in [6, Theorem 5.3] showed that the dimension cannot increase if the values of points under bonding functions are zero dimensional. The main goal of this article is to show that trivial shape is preserved under inverse limits of continua with multivalued bonding functions provided the images of points have trivial shape.

A continuum is a compact connected metric space. If \{X_i\} is a countable collection of compact metric spaces each with bounded metric \(d_i\), then \(\prod_{i=1}^{\infty} X_i\) is the countable product of the collection \{X_i\} with metric given by \(d(x, y) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i}\). For each \(j\), let \(\pi_j: \prod_{i=1}^{\infty} X_i \to X_j\) be defined by \(\pi_j((x_1, x_2, \ldots)) = x_j\) that is, \(\pi_j\) is the projection map onto the \(j\)th factor space. For each \(i\) let \(f_i: X_{i+1} \to 2^{X_i}\) be a set valued function where \(2^{X_i}\) are the closed subsets of \(X_i\). The inverse limit of the sequence of pairs \(\{(X_i, f_i)\}\), denoted \(\lim_{\leftarrow} (X_i, f_i)\), is

\[d(x, y) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i}\]

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defined to be the set of all \((x_1, x_2, \ldots)\) in \(\prod_{i=1}^{\infty} X_i\) such that \(x_i \in f_i(x_{i+1})\). The functions \(f_i\) are called bonding functions. Given topological spaces \(X\) and \(Y\), a function \(f: X \to 2^Y\) is upper semi-continuous (usc) if for each open set \(V \subseteq Y\), the set \(\{x: f(x) \subseteq V\}\) is an open set in \(X\). If \(X\) is a Hausdorff space and \(Y\) is compact, this condition is equivalent to the compactness of the graph of the function in the Cartesian product \(X \times Y\), see [4, §43, I, Theorem 4, p. 58]. A mapping or a map means a continuous function.

We start by recalling a theorem by R. B. Sher, see [1, Theorem 9.3, p. 352].

**Theorem 1.** If \(X\) and \(Y\) are finite-dimensional compact metric spaces, \(f: X \to 2^Y\) is a map such that for every \(y \in Y\), \(f^{-1}(y)\) has trivial shape then the shape of \(X\) equals the shape of \(Y\).

The theory of shape is well developed, see for example Borsuk [1]. In this paper we are only concerned with continua having trivial shape. The following are equivalent for a continuum \(X\):

1. \(X\) has trivial shape,
2. \(X\) can be written as \(X = \bigcap X_n\) where \(X_n\)'s are contractible continua,
3. \(X\) can be written as an inverse limit of contractible continua,
4. For all \(\epsilon > 0\) there exists a contractible continuum \(Y_\epsilon\) and an \(\epsilon\) map \(f_\epsilon\) from \(X\) onto \(Y_\epsilon\).

Note that a mapping \(f: X \to Y\) is cell-like if for each \(y \in Y\), \(f^{-1}(y)\) has trivial shape.

## 2. The Main Theorem

**Theorem 2.** Let \(X_1, X_2, \ldots\) be a sequence of finite dimensional continua with trivial shape, and let \(f_n: X_{n+1} \to 2^{X_n}\) be upper semi-continuous functions such that

\((1)\) \(f_n(x_{n+1})\) is a continuum with trivial shape for each \(x_{n+1}\) in \(X_{n+1}\), then \(\lim_{\leftarrow} (X_i, f_i)\) has trivial shape.

**Proof.** Following the notation of Ingram and Mahavier [3], we define \(G(f_1, f_2, \ldots, f_n) = \{x \in \prod_{i=1}^{n+1} X_i: x_i \in f_i(x_{i+1})\text{ and }1 \leq i \leq n\}\) and \(G_n = G(f_1, f_2, \ldots, f_n) \times \prod_{i>n} X_i\). Then \(G_{n+1} \subseteq G_n\) and \(\lim_{\leftarrow} (X_i, f_i) = \bigcap_{n \geq 1} G_n\), thus to show \(\lim_{\leftarrow} (X_i, f_i)\) has trivial shape it is enough to prove that each \(G_n\) has trivial shape.

To this aim define \(\pi_n: G(f_1, f_2, \ldots, f_{n-1}) \to X_n (n \geq 2)\) by \(\pi_n(x_1, x_2, \ldots, x_n) = x_n\). We show by induction that \(\pi_n\) is cell-like and \(G(f_1, f_2, \ldots, f_{n-1})\) has trivial shape.
First observe that $\pi_2^{-1}(x_2) = \{(x_1, x_2): x_1 \in f_1(x_2)\}$ is homeomorphic to $f_1(x_2)$ which has trivial shape and hence $\pi_2$ is a cell-like map and by Theorem 1, $G(f_1)$ has trivial shape.

Assume $\pi_k$ is cell-like and $G(f_1, f_2, \ldots, f_k)$ has trivial shape for all positive integers $k$ less than some positive integer $n$.

To show $\pi_n$ is cell-like, let $a \in X_n$ then $\pi_n^{-1}(a) = \{(x_1, x_2, \ldots, x_{n-1}, a): x_{n-1} \in f_n(a) \land x_i \in f_i(x_{i+1}) \textrm{ if } 1 \leq i \leq n-2\}$. Define $g: \pi_n^{-1}(a) \to f_{n-1}(a)$ by $g(x_1, x_2, \ldots, x_{n-1}, a) = x_{n-1}$. Then for $b \in f_{n-1}(a)$ one has that $g^{-1}(b) = \{(x_1, x_2, \ldots, x_{n-2}, b, a): x_i \in f_i(x_{i+1})\}$. But this is homeomorphic to $\pi_{n-1}^{-1}(b)$ which has trivial shape by our induction hypothesis. Then $g$ is a cell-like map and again by Theorem 1, $G(f_1, f_2, \ldots, f_n)$ has trivial shape. Thus all $G(f_1, f_2, \ldots, f_n)$ have trivial shape and so $\lim_{\leftarrow} (X_i, f_i) = \bigcap_{n \geq 1} G_n$ has trivial shape.

We use an example by J. L. Taylor [7] to show that the hypothesis of finite dimensionality of the factor spaces in the inverse limit cannot be discarded in Theorem 2. Taylor shows that there exists a map $g$ from a continuum $Y$ having non-trivial shape onto the Hilbert cube $Q$ and the shape of $g^{-1}(y)$ is trivial for all $y \in Y$.

Using Taylor’s map we construct an inverse sequence of Hilbert cubes with upper semi-continuous bonding functions which satisfy condition (1) of Theorem 2 and having an inverse limit with non-trivial shape. Let $g: X \to Q$ be the surjective mapping described in Taylor’s example. Embed $X$ into $Q$ and define $F: Q \to 2^Q$ by $F(q) = g^{-1}(q) \subseteq X \subseteq Q$.

Consider the inverse limit of Hilbert cubes $\{Q, F_i\}$ with $F_1 = F$ and $F_i = id$ for $i \geq 2$. Denote this inverse limit space by $X_\infty$. We show that $X_\infty$ is homeomorphic to $X$, in particular, $\pi_1$, the projection map from $X_\infty$ to the first factor space, is a homeomorphism between $X_\infty$ and $X$.

Clearly $\pi_1$ maps $X_\infty$ onto $X$. We need to see that $\pi_1$ is injective. Let $x$ and $y$ be distinct points in $X_\infty$. Then there is an integer $n$ such that $x_n = \pi_n(x) \neq \pi_n(y) = y_n$. If $n = 1$ then we are done. If $n > 2$ then $x_n = x_2$ and $y_n = y_2$ and thus $x_2 \neq y_2$. Let $x_1 \in F_1(x_2)$ and $y_1 \in F_1(y_2)$ then $g(x_1) = x_2 \neq y_2 = g(y_1)$. Thus $x_1 \neq y_1$ showing that $\pi_1$ is injective as claimed.

3. Examples

**Example 1.** Let $f: [0, 1] \to 2^{[0, 1]}$ be defined as $f(x) = x$ if $x \in [0, 1)$ and $f(1) = [0, 1]$. Since the graph of $f$ is closed, by [3, Theorem 2.1], $f$ is upper semi-continuous. Then the hypothesis of Theorem 2 are satisfied so $\lim_{\leftarrow} ([0, 1], f)$ must have trivial shape. In fact, $\lim_{\leftarrow} ([0, 1], f)$ is the union of the countably
many arcs \( \{(x, x, \ldots, x, 1, 1, \ldots) : \ x \in [0, 1]\} \). These arcs intersect in the one point \((1, 1, 1, \ldots)\) hence \( \lim_{\xi} ([0, 1], f) \) is a harmonic fan.

In our next example we consider what seems like a very similar function but the associated inverse limit space is much more complex. Theorem 2 can be applied here to help understand the structure of the inverse limit space.

**Example 2.** Let \( f : [0, 1] \to 2^{[0, 1]} \) be defined as \( f(x) = 1 - x \) if \( x \in (0, 1] \) and \( f(0) = [0, 1] \). The inverse limit space \( \lim_{\xi} ([0, 1], f) \) contains the arc \( A_0 = \{(x, 1 - x, x, 1 - x, \ldots) : x \in [0, 1]\} \). The end points \((1, 0, 1, 0, \ldots)\) and \((0, 1, 0, 1, \ldots)\) of this arc are ramification points of countable order in \( \lim_{\xi} (X, f) \). To see this consider first the arc \( A_1 = \{(x, 0, 1, 0, \ldots) : x \in [0, 1]\} \). \( A_0 \) and \( A_1 \) have the point \((1, 0, 1, 0, \ldots)\) in common. The arc \( A_2 = \{(1 - x, x, 1 - x, 0, 1, 0, \ldots) : x \in [0, 1]\} \) joins \((1, 0, 1, 0, \ldots)\) to \((0, 1, 0, 1, 0, 1, \ldots)\) which in turn is joined by the arc \( \{(1 - x, 0, 0, 0, 1, 0, \ldots) : x \in [0, 1]\} \) to the point \((0, 0, 0, 0, 1, 0, \ldots)\). Points with \( n (n > 1) \) leading zeroes and then alternating 1’s and 0’s are endpoints of the continuum. The same process can be carried out at the other endpoint of \( A_0 \) except the endpoints of these arcs will have an odd number of leading zeroes.

The point \((0, 0, 0, \ldots)\) is the endpoint of the arc \( \{(x, 0, 0, \ldots) : x \in [0, 1]\} \) which joins it to the point \((1, 0, 0, \ldots)\). This point can be joined by an arc to \((0, 1, 0, 0, \ldots)\).

Continuing we see that \((0, 0, 0, \ldots)\) is in fact an endpoint of a ray which approaches the arc \( A_0 \).

The endpoints of the arcs \( A_n \) approach the point \((0, 0, 0, \ldots)\) so one might wonder if the inverse limit space of this example is unicoherent. Note that if \( x \) is in \( \lim_{\xi} ([0, 1], f) \) and the \( n \)-th coordinate of \( x \) equals zero then \( \pi_n(x) \in \{0, 1\} \) for all \( m > n \). Let \( X_n = \{x \in \lim_{\xi} ([0, 1], f) : \ \pi_n(x) \neq 0 \text{ and } \pi_{n+1}(x) = 0\} \) then \( X_n \) is a Cantor set cross an interval, thus one dimensional. If \( X_\infty = \{x \in \lim_{\xi} ([0, 1], f) : \ \pi_n(x) \neq 0 \text{ for all } n\} \) then \( X_\infty \) is an interval and \( \lim_{\xi} ([0, 1], f) = \bigcup_{n \geq 0} X_n \) so \( \lim_{\xi} ([0, 1], f) \) is one dimensional as a countable union of closed one-dimensional sets. The one-dimensionality of the inverse limit space also follows from Van Nall [6, Theorem 5.4]. It follows from Case-Chamberlain characterization of tree-like continua (see [2, Theorem 1, p. 74]) that every one-dimensional continuum with trivial shape is tree-like, and therefore \( \lim_{\xi} ([0, 1], f) \) is hereditarily unicoherent.

**Example 3.** Let us look at the inverse limit of intervals \([-1, 1]\) with the only bonding function \( f : [-1, 1] \to 2^{[-1, 1]} \) defined by \( f(-1) = f(1) = [-1, 1] \) and \( f(x) = 0 \) for \( x \in (-1, 1) \). Thus the graph of \( f \) looks like capital letter H. According
to Theorem 2, the inverse limit is a continuum of trivial shape. We will show that it is one-dimensional, so it is a tree-like continuum and that it is arcwise connected, so it is a dendroid.

To see that \( X = \lim_{\leftarrow} \left( [-1,1],f \right) \) is one-dimensional, it is enough to represent \( X \) as the countable union of closed 0-dimensional sets. To this aim define \( P_n = \left\{ (x_1,x_2,\ldots) \in X : x_1 = \cdots = x_{n-1} = 0 \text{ and } x_{n+1},x_{n+2},\ldots \in \{-1,1\} \right\} \) and note that \( P_n \) is homeomorphic to the Cartesian product of \([-1,1]\) (the \( n \)-th coordinate) and the Cantor set (-1's and 1's on coordinates \( n+1, n+2, \ldots \)), thus \( P_n \) is zero-dimensional and closed. Finally, if we denote by \( p \) the point \( p = (0,0,\ldots) \), then \( X = \{p\} \cup \bigcup \{P_n : n \in \{1,2,\ldots\}\} \).

To see that \( X \) is arcwise connected observe that each point of \( P_n \) can be joined by an arc, varying the \( n \)-th coordinate, to a point of \( P_{n+1} \), then, varying the \( n+1 \)-st coordinate to a point of \( P_{n+2} \), etc. Then the union of those arcs is a ray, i.e. homeomorphic image of \([0,\infty)\), and since \( \lim_{n \to \infty} P_n = \{p\} \), the closure of the ray is an arc containing \( p \). Arcwise connected tree-like continua are dendroids, so \( X \) is a dendroid.

We will show that \( X \) is in fact a smooth dendroid, because it is a monotone relative to the top image of the Cantor fan, see [5, Corollary 2.10, p. 722].

Represent the Cantor set \( C \) as the infinite Cartesian product \( C = \{-1,1\}^\infty \) and denote by \( F \) the Cantor fan \( C \times [0,1]/C \times \{0\} \); its top \( C \times \{0\} \) will be denoted by \( q \). For every \( x = (x_1,x_2,\ldots) \in X \) define \( d(x) = \sum_{n=1}^{\infty} 2^{-n} |x_n| \) and note that \( d \) maps \( X \) onto \([0,1] \). Denote by \( E \) the set \( E = \{(x_1,x_2,\ldots) \in X : x_n \in \{-1,1\} \text{ for } n \in \{1,2,\ldots\}\} \) and observe that \( E = d^{-1}(1) \) and that it is homeomorphic (in fact it is identical) to the Cantor set \( C \); let \( h : C \to E \) be the homeomorphism. For every \( e \in E \) the mapping \( d \) restricted to the arc \( pe \) is a homeomorphism of \( pe \) onto \([0,1] \) and thus we may define a mapping \( g : F \to X \) by letting \( g(e,t) \) be the only point \( x \) of the arc \( h(e)p \) that satisfies \( d(x) = t \). One can see that such defined mapping \( g \) satisfies the condition \( g(qx) = g(q)g(x) \) for every \( x \in F \), therefore \( g \) is a mapping monotone relative to the top \( q \) by [5, Corollary 2.11, p. 722], and \( X \), as the image of \( F \) under \( g \), is a smooth dendroid by [5, Corollary 2.11, p. 722]. One can also see that each arc component of \( X \setminus \{p\} \) is dense in \( X \) and that each ramification point different than \( p \) is of order three.

References


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