A WEAKER FORM OF THE PROPERTY OF KELLEY

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Abstract. A weaker form of the property of Kelley for metric continua is defined and studied. Its mapping properties and connections with related concepts are investigated. Several results known for continua having the property of Kelley are generalized to continua satisfying the weaker condition.

1. Introduction

A metric continuum X is said to have the *property of Kelley* provided that for each point $x \in X$, for each subcontinuum K of X containing x and for each sequence of points x_n converging to x there exists a sequence of subcontinua K_n of X containing x_n and converging to the continuum K (see e.g. [18, Definition 16.10, p. 538]).

The property, introduced by J. L. Kelley as property 3.2 in [12, p. 26], has been used there to study hyperspaces, in particular their contractibility (see e.g. Chapter 16 of [18], where

1991 *Mathematics Subject Classification.* 54B20, 54C10, 54E40, 54F15, 54F50, 54F65.

Key words and phrases. confluent, continuum, hyperspace, mapping, maximal limit continuum, property of Kelley, semi-confluent, semi-Kelley.

references for further results in this area are given). Now the property, which has been recognized as an important tool in investigation of various properties of continua, is interesting by its own right, and has numerous applications to continuum theory. Many of them are not related to hyperspaces. A pointed version of this property has been introduced by Wardle in [20, p. 291], where it is shown that homogeneous continua have the property of Kelley. This result has been extended to openly homogeneous continua in [2, Statement, p. 380], and Kato has proved in [9] that it cannot be enlarged to continua that are homogeneous with respect to confluent mappings (introduced in [1]). Very recently the second named author has generalized the property of Kelley in [8] to the non-metric case, and constructed an example showing that, unlike for metric continua, the homogeneity of non-metric ones does not imply the property of Kelley. A rather narrow class of continua, namely fans, having the property of Kelley has been characterized by the authors in [3] and [4].

In [10] Kato defined a stronger version of the property of Kelley and showed that if a continuum X has this stronger property, then the hyperspace $C(X)$ of all nonempty subcontinua of X, as well as all Whitney continua in $C(X)$ have the property of Kelley. In the present paper we are going in the opposite direction: we introduce a weaker version of the property of Kelley and we investigate various consequences of this new notion.

The paper consists of five chapters. After Introduction, basic concepts used in the paper are collected in Preliminaries. In the third chapter notions of maximal and strong maximal limit continua are introduced, and their properties are studied. In particular, using these concepts, a characterization of continua having the property of Kelley is obtained in Theorem 3.11. The weaker form of the property of Kelley (called semi-Kelley) also is introduced in this chapter. In Chapter 4 properties of products and hyperspaces of semi-Kelley continua are considered. In particular in Theorems 4.1 and 4.5 we improve two results of Wardle from [20] by weakening assumptions from continua having the property of Kelley to semi-Kelley continua. A similar improvement is shown in Theorem 4.9 for a result of Moon, Hur and Rhee from [16]. Mapping properties of the introduced weaker form of the property of Kelley are studied in Chapter 5. We focused our attention on confluent, semi-confluent and weakly confluent mappings. Some analogs as well as generalizations of results known for continua having the property of Kelley are obtained for the weaker form of this property. In particular it is shown that the uniform limit of confluent mappings onto a semi-Kelley continuum is semi-confluent. A number of examples are presented and open problems are posed.

2. Preliminaries

All spaces considered in this paper are assumed to be metric. A *mapping* means a continuous function.

Given a (metric) space X we denote by d_X the metric on X, and by $B_X(p, \varepsilon)$ the (open) ball in X centered at a point $p \in X$ and having the radius ε. Given a subset $A \subset X$, we put $N_X(A, \varepsilon) = \bigcup\{B_X(a, \varepsilon) : a \in A\}$. The symbol $\mathbb{N}$ stands for the set of all positive integers, and $\mathbb{R}$ denotes the space of real numbers.

A *continuum* means a compact connected space. Given a continuum X with a metric d, we let 2^X to denote the hyperspace of all nonempty closed subsets of X equipped with the Hausdorff metric H defined by

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\},\\ \sup\{d(b, A) : b \in B\}\} \quad \text{for } A, B \in 2^X$$

(see e.g. [18, (0.1), p. 1 and (0.12), p. 10]). If $H(A, A_n)$ tends to zero as n tends to infinity, we write $A = \operatorname{Lim} A_n$. For an element $E \in 2^X$ and a subset $\mathcal{S} \subset 2^X$ the symbol $H(E, \mathcal{S})$ is defined by $H(E, \mathcal{S}) = \inf\{H(E, F) : F \in \mathcal{S}\}$. Further, we denote by $C(X)$ the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X, and by $F_1(X)$ the hyperspace of singletons. The symbol $C^2(X)$ denotes $C(C(X))$, and $\mathcal{H}$ means the Hausdorff metric in $C^2(X)$. The symbol $\cup$ stands for the union mapping that assigns to a subset of $C(X)$ the union of all its elements (see [18, Lemmas 1.48 and 1.49, p. 100 and 102], respectively).

An *order arc* in the hyperspace 2^X is an arc such that for every two its elements A and B one is contained in the other, i.e., either $A \subset B$ or $B \subset A$. See subchapter (I.A) of [18, p. 56] for an information about this concept.

Given a continuum X, let $\Lambda(X)$ denote the hyperspace either 2^X or $C(X)$. A mapping $\mu : \Lambda(X) \to [0, \infty)$ is called a *Whitney map for* $\Lambda(X)$ provided that $\mu(\{x\}) = 0$ for each point $x \in X$, and that $\mu(A) < \mu(B)$ for every two elements $A, B \in \Lambda(X)$ such that $A \subsetneq B$. We say that a Whitney map μ for $C(X)$ is *normalized* provided that $\mu(X) = 1$; then $\mu : C(X) \to [0, 1]$. It is known that each Whitney map for $C(X)$ is monotone, while it does not have to be such for 2^X. For each $t \in [0, \mu(X)]$ the continua $\mu^{-1}(t)$ are called *Whitney levels for* $\Lambda(X)$. Thus for $A \in C(X)$ the intersection $\mu^{-1}(t) \cap C(A) = (\mu|C(A))^{-1}(t)$ is a Whitney level, so it is a continuum. The reader is referred to Nadler's book [18] for more information about hyperspaces and related concepts.

A continuum X is said to be *unicoherent* provided that the intersection of every two of its subcontinua whose union is X is connected; *hereditarily unicoherent* provided that each of its subcontinua is unicoherent. A hereditarily unicoherent and arcwise connected continuum is called a *dendroid*. A point x of a dendroid X is called an *end point* of X provided that x is an end point of any arc in X that contains x. A point of a dendroid X is called a *ramification point* of X provided that it is a vertex of a simple triod contained in X, i.e., if there are three arcs xa, xb and xc in X having x as the only point of the intersection of any two of them. By a *fan* we understand a dendroid having exactly one ramification point, usually denoted by v and called the *top* of the fan. Names of the following three fans will frequently be used to illustrate various relations between discussed concepts in the paper. So, let us define them.

Given two points p and q in the plane we let pq denote the straight line segment with end points p and q. In the polar coordinates (ρ, ϕ) in the plane with pole $v = (0, 0)$, consider for each $n \in \mathbb{N}$

$$a_0 = (1, 0), \quad a_n = (1, 1/n), \quad b_n = (1/2, 1/n) \quad \text{and} \quad c = (2, 0).$$

Then the union

$$X_1 = va_0 \cup \bigcup \{va_n : n \in \mathbb{N}\} \tag{2.1}$$

is called the *harmonic fan*. It is homeomorphic to the cone over the closure of the harmonic sequence $\{0\} \cup \{1/n : n \in \mathbb{N}\}$.

The union

$$X_2 = vc \cup \bigcup \{va_n : n \in \mathbb{N}\} = X_1 \cup a_0 c \tag{2.2}$$

is called the *harmonic prolonged fan*. It serves as a typical example of a continuum that does not have the property of Kelley.

Out third example is defined as the union

$$X_3 = va_0 \cup \bigcup \{(va_{2n} \cup a_{2n} b_{2n+1}) : n \in \mathbb{N}\}, \tag{2.3}$$

and it is called the *harmonic hooked fan* (see [5, (23) and Fig. 7, p. 69]).

3. Maximal limit continua

The following form of the property of Kelley (see Introduction) will be used in the sequel. Its proof is left to the reader.

Proposition 3.1. *A continuum X has the property of Kelley if and only if for each point $x \in X$, for each subcontinuum K of X containing x and for each sequence of points x_n converging to x there exists a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ and a sequence of subcontinua K_k of X containing x_{n_k} and converging to the continuum K.*

The next two auxiliary concepts play an important role in investigation of the weaker form of the property of Kelley in the paper.

Definition 3.2. Let K be a subcontinuum of a continuum X. A continuum $M \subset K$ is called a *maximal limit continuum in* K provided that there is a sequence of subcontinua M_n of X converging to M such that for each convergent sequence of subcontinua M'_n of X with $M_n \subset M'_n$ for each $n \in \mathbb{N}$ and $\operatorname{Lim} M'_n = M' \subset K$ we have $M' = M$.

Definition 3.3. Let K be a subcontinuum of a continuum X. A continuum $M \subset K$ is called a *strong maximal limit continuum in* K provided that there is a sequence of subcontinua M_n of X converging to M such that for each subsequence $\{M_{n_k}\}$ of the sequence $\{M_n\}$ and for each convergent sequence of subcontinua M'_k with $M_{n_k} \subset M'_k$ for each $k \in \mathbb{N}$ and $\operatorname{Lim} M'_k = M' \subset K$ we have $M' = M$.

The next statement is obvious.

Statement 3.4. *For each continuum X and its subcontinuum K every strong maximal limit continuum in K is a maximal limit continuum in K.*

An example is presented below that shows the difference between the two concepts: the inverse implication to that of Statement 3.4 is not true.

Example 3.5. *There is a continuum X, its subcontinuum K and a continuum $M \subset K$ which is maximal limit continuum in K while not strong maximal limit continuum in K.*

Proof: In the Euclidean plane let pq denote the straight line segment joining p and q. Put (in the Cartesian coordinates) $v = (0,0)$, $a = (0,1)$, $b = (-1,0)$, $c = (1,0)$, and for each $n \in \mathbb{N}$, let $b_n = (-1, 1/n)$, $c_n = (1, 1/n)$, $p_n = (-1/n, 1/n)$ and $q_n = (1/n, 1/n)$. Define

$$X = va \cup vb \cup vc \cup \bigcup\{ap_n \cup p_n b_n \cup aq_n \cup q_n c_n : n \in \mathbb{N}\}.$$

Further, let $a' = (0, 1/2)$, $b' = (-1/2, 0)$ and $c' = (1/2, 0)$ be the mid points of the segments va, vb and vc, respectively. Then $M = va'$ is a maximal limit continuum in $K = va' \cup vb' \cup vc'$ by Definition 3.2, and it is not any strong maximal limit continuum in K, because if p'_n and q'_n stand for mid points of the segments ap_n and aq_n respectively, then defining $M_{2n-1} = p'_n p_n$ and $M_{2n} = q'_n q_n$ we get a sequence of continua M_n converging to M such that for each convergent sequence of continua M'_n with $M_n \subset M'_n$ if $\operatorname{Lim} M'_n \subset K$, then $\operatorname{Lim} M'_n = M$, while the implication does not hold if we consider the subsequence $\{M_{2n-1}\}$, for example.

Note that each subcontinuum K of X is a strong maximal limit continuum (thus, by Statement 3.4, a maximal limit continuum) in itself. Further, the next observation is a consequence of the definitions.

Observation 3.6. *Let M, L and K be subcontinua of a continuum X with $M \subset L \subset K$. If M is a (strong) maximal limit continuum in K, then M is a (strong) maximal limit continuum in L.*

The following concept will be useful in the sequel. For a continuum X, and a sequence of continua M_n converging to a continuum $M \subset X$ define $\mathcal{M}(\{M_n\})$ as the family of all subcontinua A of X with $M \subset A$ such that there exists a subsequence $\{M_{n_k}\}$ of the sequence $\{M_n\}$ and there is a convergent sequence of subcontinua A_k of X such that $M_{n_k} \subset A_k$ for each $k \in \mathbb{N}$, and $\operatorname{Lim} A_k = A$. In the next statement an equivalent form of the definition of the family $\mathcal{M}(\{M_n\})$ is formulated.

Statement 3.7. *Let a sequence of continua M_n converging to a continuum M in a continuum X be fixed. Then a continuum A belongs to $\mathcal{M}(\{M_n\})$ if and only if for each $\varepsilon > 0$ and for each $n \in \mathbb{N}$ there is an integer $m > n$ and a continuum B_m containing M_m such that $H(A, B_m) < \varepsilon$.*

Proposition 3.8. *For each sequence of continua M_n converging to a continuum $M \subset X$ the family $\mathcal{M}(\{M_n\})$ is closed in the hyperspace $C(X)$.*

Proof: Let us fix a sequence of continua M_n converging to a continuum M, and, for shortness, write $\mathcal{M}$ in place of $\mathcal{M}(\{M_n\})$. Consider a convergent sequence of continua $A^n \in \mathcal{M}$, and let $A = \operatorname{Lim} A^n$. We have to show that the continuum A is a member of $\mathcal{M}$. We apply induction. Since $A^1 \in \mathcal{M}$, by Statement 3.7 there is an index $n_1 \in \mathbb{N}$ and a continuum B_{n_1} such that $M_{n_1} \subset B_{n_1}$ and $H(A^1, B_{n_1}) < 1$. In general, Statement 3.7 guarantees, for each $k \in \mathbb{N}$, the existence of an index n_k which is greater than the previous one, n_{k-1}, and a continuum B_{n_k} such that $M_{n_k} \subset B_{n_k}$ and $H(A^k, B_{n_k}) < 1/k$. Thus we have $\operatorname{Lim} B_{n_k} = \operatorname{Lim} A^k = A$, whence $A \in \mathcal{M}$ again by Statement 3.7. Therefore $\mathcal{M}$ is closed in $C(X)$.

Let K be a nonempty subcontinuum of a continuum X, and let a sequence of continua M_n converging to a continuum $M \subset K$ be given. Since $C(K) \cap \mathcal{M}(\{M_n\})$ is closed in $C(K)$ according to the above statement, it follows from Kuratowski-Zorn Lemma that there is a maximal (with respect to inclusion) element in $C(K) \cap \mathcal{M}(\{x_n\})$.

Proposition 3.9. *For each subcontinuum K of a continuum X and for each sequence of continua M_n converging to a continuum $M \subset K \subset X$ a maximal element S in $C(K) \cap \mathcal{M}(\{M_n\})$ is a strong maximal limit continuum in K.*

Proof: Let S be a maximal element in $C(K) \cap \mathcal{M}(\{M_n\})$. By the definition of $\mathcal{M}(\{M_n\})$ we see that there exists a subsequence $\{M_{n_k}\}$ of the sequence $\{M_n\}$ and there is a sequence of continua S_k such that $M_{n_k} \subset S_k$ and $S = \operatorname{Lim} S_k$. Choose an arbitrary subsequence $\{n_{k_i}\}$ of the sequence $\{n_k\}$, and let for each $i \in \mathbb{N}$ a continuum S'_i be such that $S_{k_i} \subset S'_i$ and that $S' = \operatorname{Lim} S'_i \subset K$. Then $S' \in C(K) \cap \mathcal{M}(\{M_n\})$ and $S \subset S'$, whence $S' = S$ by maximality of S. Therefore S is a strong maximal limit continuum in K, so the proof is complete.

As a corollary to Proposition 3.9 we have the next statement.

Statement 3.10. *Let K be a nonempty subcontinuum of a continuum X. Then for each maximal limit continuum M in K there exists a strong maximal limit continuum S in K such that $M \subset S$.*

The next result characterizes continua having the property of Kelley in terms of the above introduced concepts.

Theorem 3.11. *The following conditions are equivalent for a continuum X:*

(3.12) *X has the property of Kelley;*

(3.13) *for each subcontinuum K of X, the only maximal limit continuum in K is K itself;*

(3.14) *for each subcontinuum K of X, the only strong maximal limit continuum in K is K itself.*

Proof: The implication from the property of Kelley to (3.13) is obvious. Assume (3.13), and let S be a strong maximal limit continuum in K. By Statement 3.4 S a maximal limit continuum in K, so $S = K$ by (3.13), and thus (3.14) follows. If (3.14) is assumed, then let a point $x \in X$, a subcontinuum K of X containing x and a sequence of points x_n converging to x be given. Let S be a strong maximal limit continuum in K. By assumption we have $S = K$, and by the definition of $\mathcal{M}(\{\{x_n\}\})$ it follows that $K \in \mathcal{M}(\{\{x_n\}\})$, which means that there is a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ and a sequence of continua K_k such that $x_{n_k} \in K_k$ for each $k \in \mathbb{N}$ and that $K = \operatorname{Lim} K_k$. Therefore X has the property of Kelley according to Proposition 3.1. The proof is finished.

Proposition 3.15. *If a continuum X does not have the property of Kelley, then there exists a subcontinuum K of X and a strong maximal limit continuum M in K such that M is a nondegenerate proper subset of K.*

Proof: By equivalence of (3.12) and (3.14) of Theorem 3.11 we infer that there is a subcontinuum K_0 of X and a strong maximal limit continuum M_0 in K_0 such that $M_0 \neq K_0$. If M_0 is nondegenerate, we are done. So assume that M_0 is a singleton $\{x\}$. Take a sequence of subcontinua M_n of X as in Definition 3.3 of a strong maximal limit continuum. For each $n \in \mathbb{N}$ choose a point $x_n \in M_n$. Then the sequence of points x_n is convergent, and it has x as its limit. Let $\mu : C(X) \to [0,1]$ be a normalized Whitney map for $C(X)$. Define a function $F : X \times [0,1] \to C(X)$ by

$$F(x,t) = \bigcup \{A \in C(X) : x \in A \quad \text{and} \quad \mu(A) \leq t\}.$$

Since there is an order arc from the singleton $\{x\}$ to X, and since this order arc intersects the Whitney level $\mu^{-1}(t)$, hence there exists $A \subset F(x,t)$ such that $\mu(A) = t$. Thus $\mu(F(x,t)) \geq t$.

Now take the family $\mathcal{M}(\{\{x_n\}\})$ and observe that since $M_0 = \{x\}$ is a strong maximal limit continuum in K_0, it follows that K_0 is not a member of this family. Then by closedness of $\mathcal{M}(\{\{x_n\}\})$ in $C(X)$ (see Proposition 3.8) there exists a neighborhood of K_0 in $C(X)$ disjoint with $\mathcal{M}(\{\{x_n\}\})$, whence it follows that there is a number $t_0 \in (0, \mu(K_0)) \subset [0,1]$ such that for each $A \in C(X)$ with $x \in A$ the condition $\mu(A) < t_0$ implies that $K_0 \cup A \notin \mathcal{M}(\{\{x_n\}\})$. Choose a number $t \in [0,1]$ so that $\mu(F(y,t)) \leq t_0$ for each point $y \in X$. Consider a subsequence of points x_{n_k} that the sequence $F(x_{n_k}, t)$ is convergent. For shortness put $F_k = F(x_{n_k}, t)$ and let $F = \operatorname{Lim} F_k$. Note that $F \in \mathcal{M}(\{\{x_n\}\})$. Define $K = K_0 \cup F$. Since the family $C(K) \cap \mathcal{M}(\{\{x_n\}\})$ is a closed subset of $C(X)$ (see Proposition 3.8) there exists its maximal (with respect to inclusion) element M containing F. Since $\mu(F_k) \geq t$, we have $\mu(F) \geq t$, whence it follows that M is not degenerate. Further, $t < \mu(K_0)$ by the choice of t, which implies that M is a proper subset of K. Finally it follows from Proposition 3.9 that M is a strong maximal limit continuum in K. The proof is then complete.

Let us introduce the following concept.

Definition 3.16. A continuum X is said to be *semi-Kelley* provided that for each subcontinuum K of X and for every two maximal limit continua M_1 and M_2 in K either $M_1 \subset M_2$ or $M_2 \subset M_1$.

The next statement is obvious.

Statement 3.17. *If a continuum has the property of Kelley, then it is semi-Kelley.*

The inverse implication to that in Statement 3.17 does not hold, because the harmonic prolonged fan (2.2) and the harmonic hooked fan (2.3) are semi-Kelley, while they do not have the property of Kelley.

We conclude this section with showing that in Definition 3.16 of a semi-Kelley continuum the condition can be applied to strong maximal limit continua only.

Theorem 3.18. *For each continuum X the following are equivalent.*

(3.19) *X is a semi-Kelley continuum;*

(3.20) *for each subcontinuum K of X and for every two strong maximal limit continua M_1 and M_2 in K either $M_1 \subset M_2$ or $M_2 \subset M_1$.*

Proof: The implication from (3.19) to (3.20) follows from Statement 3.4. To see the opposite implication, assume (3.20) and suppose, on the contrary, that (3.19) does not hold. It means that there is a subcontinuum K of X and there are two maximal limit continua M and M' in K such that $M \setminus M' \neq \emptyset \neq M' \setminus M$. By the Kuratowski-Zorn lemma there exists in K a continuum L which is irreducible with respect to containing $M \cup M'$, i.e., such that $M \cup M' \subset L$ and no proper subcontinuum of L contains this union.

Let S and S' be arbitrary strong maximal limit continua in L which contain M and M', respectively. The existence of such continua follows from Statement 3.10. Thus by (3.20) either $S \subset S'$ or $S' \subset S$. If $S \subset S'$, then $M \cup M' \subset S'$, whence $S' = L$ by irreducibility of L. Similarly, $S' \subset S$ implies $S = L$.

Consequently, we have shown that either each strong maximal limit continuum in L containing M is equal to L, or each strong maximal limit continuum in L containing M' is equal to L.

Assume the former possibility, and let $\{M_n\}$ be a sequence of continua converging to the continuum M as in the definition of the maximal limit continuum M in K. Then for each subsequence $\{M_{n_k}\}$ of the sequence $\{M_n\}$ we have $L \in C(L) \cap \mathcal{M}(\{M_{n_k}\})$. This means that

(3.21) for each subsequence $\{M_{n_k}\}$ of the sequence $\{M_n\}$ there is a subsequence $\{M_{n_{k_i}}\}$ of $\{M_{n_k}\}$ and there are continua S_i with $M_{n_{k_i}} \subset S_i$ such that $L = \operatorname{Lim} S_i$.

We will show that for each $n \in \mathbb{N}$ there is a continuum L_n containing M_n with $L = \operatorname{Lim} L_n$, contrary to the definition of M as a maximal limit continuum in K. To this aim for each $n \in \mathbb{N}$ define L_n as an element of the family $\mathcal{F}_n = \{P \in C(X) : M_n \subset P\}$ such that

$$H(L_n, L) = \min\{H(P, L) : P \in \mathcal{F}_n\}.$$

Then (3.21) implies that $\operatorname{Lim} L_n = L$. The proof is complete.

4. Products and hyperspaces

It is known that if the Cartesian product of two continua has the property of Kelley, then each factor continuum has the property of Kelley, too, [20, Corollary 4.6, p. 297]. This result can be strengthened by assuming that the product is a semi-Kelley continuum.

Theorem 4.1. *If the Cartesian product of two nondegenerate continua is semi-Kelley, then each factor continuum has the property of Kelley.*

Proof: Let two nondegenerate continua X and Y be given such that their product $X \times Y$ is semi-Kelley. Suppose that X does not have the property of Kelley. Then, by condition (3.13) of Theorem 3.11, there exists a subcontinuum K of X and a maximal limit continuum M in K such that $M \subsetneq K$.

Let $a \in K \setminus M$, and $p, q \in Y$ be two distinct points. Define a continuum $L \subset X \times Y$ by $L = (K \times \{p, q\}) \cup (\{a\} \times Y)$. We will show that $M \times \{p\}$ and $M \times \{q\}$ are disjoint maximal limit continua in L. Take a sequence of subcontinua M_n of X converging to M as in the definition of a maximal limit continuum in K. Then the sequence $M_n \times \{p\}$ tends to $M \times \{p\}$. For each $n \in \mathbb{N}$ take $M'_n \subset X \times Y$ such that $M_n \times \{p\} \subset M'_n$ and $M' = \operatorname{Lim} M'_n \subset L$. We will show that M' is equal to $M \times \{p\}$. Suppose the contrary. Then there is a point $x \in K \setminus M$ such that $(x, p) \in M'$. Let π_1 denote the projection of the product on the first factor. Hence the sequence $\pi_1(M'_n)$ tends to $\pi_1(M') \subset K$ and we have $M \subsetneq \pi_1(M')$, contrary to the definition of a maximal limit continuum in K. Since for $M \times \{q\}$ the argument is the same, the proof is complete.

Corollary 4.2. (Wardle) *If the Cartesian product of two continua has the property of Kelley, then each factor continuum has the property of Kelley.*

An example is constructed in [20, Example 4.7, p. 297] (see also [18, Example (16.35), p. 558]) of a continuum Y having the property of Kelley whose Cartesian square $Y \times Y$ does not have this property. Later it was shown in [6, Example, p. 458] that for the same Y the hyperspace $C(Y)$ has the property of Kelley, while 2^Y does not have the property (compare also [7, Example, p. 8]). Below Y is used as a part of a continuum X to show that similar phenomena hold for semi-Kelley continua. Thus the inverse implication to that of Theorem 4.1 does not hold.

Example 4.3. *There exists a continuum X having the property of Kelley such that $X \times X$ and 2^X are not semi-Kelley.*

Proof: Let S be the unit circle in the (complex) plane $\mathbb{R}^2$. Put $\mathbb{H} = [1, \infty) \subset \mathbb{R}$, and define mappings $f, g : \mathbb{H} \to \mathbb{R}^2$ by

$$f(t) = (1 + \frac{1}{t})e^{it} \quad \text{and} \quad g(t) = (1 - \frac{1}{t})e^{-it} \quad \text{for } t \in \mathbb{H}.$$

Let $L = g(\mathbb{H})$ and $M = f(\mathbb{H})$. Then $Y = L \cup S \cup M$ is a continuum in the plane $\mathbb{R}^2$ (compare [20, Example 4.7, p. 297] and [6, Example, p. 458]). Take two copies $Y_0 = L_0 \cup S_0 \cup M_0$ and $Y_1 = L_1 \cup S_1 \cup M_1$ of Y, identify the end points of the outer spirals M_0 and M_1, and let X be the resulting continuum. Then X has the property of Kelley. To show that $X \times X$ is not semi-Kelley recall that the following fact is shown in [20, Example 4.7, p. 297] in order to prove $Y \times Y$ does not have the property of Kelley.

Fact 1. Let $\Delta Y = \{(y, y) : y \in Y\} \subset Y \times Y$. If a continuum $K \subset Y \times Y$ satisfies $K \subset N_{Y\times Y}(\Delta Y, \pi)$ and if there is a point $(x_0, y_0) \in K$ with $x_0 \in M$ and $y_0 \in L$, then every point $(x, y) \in K$ satisfies $x \in M$ and $y \in L$.

In other words, $\Delta S = \{(y, y) : y \in S\}$ is a maximal limit continuum in ΔY. This implies that ΔS_0 and ΔS_1 are disjoint maximal limit continua in ΔX. Thus $X \times X$ is not semi-Kelley.

The proof that 2^X is not semi-Kelley is very similar. The following fact is a consequence of an argument given in the proof that 2^Y does not have the property of Kelley in [6, Example, p. 458].

Fact 2. Put $F_1(Y) = \{\{y\} : y \in Y\}$. If a continuum $\mathcal{K} \subset 2^Y$ satisfies $\mathcal{K} \subset N_{2^Y}(F_1(Y), \pi)$ and if there is a point $(x_0, y_0) \in \mathcal{K}$ with $x_0 \in M$ and $y_0 \in L$, then for every $K \in \mathcal{K}$ we have $K \cap M \neq \emptyset \neq K \cap L$.

In other words, $F_1(S)$ is a maximal limit continuum in $F_1(Y)$. This implies that $F_1(S_0)$ and $F_1(S_1)$ are disjoint maximal limit continua in $F_1(X)$. Thus 2^X is not semi-Kelley. The proof is complete.

In [11, Problem 3.4, p. 1148] Kato asks if the property of Kelley of a continuum X implies that the product $X \times [0, 1]$ also has the property? In connection with Theorem 4.1 and Example 4.3 one can extend Kato's question to the following.

Question 4.4. Is it true that if a continuum X has the property of Kelley, then the Cartesian product $X \times [0, 1]$ is semi-Kelley?

In [20, Theorem 2.8, p. 294] Wardle has shown that if the hyperspace $C(X)$ of a continuum X has the property of Kelley, then X also has the property. Our next theorem shows that the assumption in that implication can be weakened.

Theorem 4.5. *Let a continuum X be given. If the hyperspace $C(X)$ is a semi-Kelley continuum, then X has the property of Kelley.*

Proof: Suppose that X does not have the property of Kelley. Then by Proposition 3.15 there exists a subcontinuum K of X and a (strong) maximal limit continuum M in K such that M is a nondegenerate proper subset of K. Take a sequence of subcontinua M_n of X converging to M as in the definition of a maximal limit continuum in K, and let $a \in K \setminus M$. Let μ be a Whitney map for $C(X)$. Since M is nondegenerate, $\mu(M) > 0$. Choose a number $t \in (0, \mu(M))$. Let $\mathcal{A}$ be an order arc in $C(X)$ such that $\{a\} \in \mathcal{A}$ and $\mu(\cup\mathcal{A}) = t$. Define

$$\mathcal{K} = F_1(K) \cup \mathcal{A} \cup (C(K) \cap \mu^{-1}(t)).$$

To finish the proof we will show, applying condition (3.20) of Theorem 3.18, that $\mathcal{K}$ contains two disjoint maximal limit continua.

Claim 1. $F_1(M)$ is a maximal limit continuum in $\mathcal{K}$.
Note that $F_1(M_n)$ are continua in $C(X)$ tending to $F_1(M)$. For each $n \in \mathbb{N}$ let $\mathcal{M}_n$ be a subcontinuum of $C(X)$ such that $F_1(M_n) \subset \mathcal{M}_k$ and that the sequence $\{\mathcal{M}_n\}$ converges to a continuum $\mathcal{M}$ in $\mathcal{K}$. Obviously $F_1(M) \subset \mathcal{M}$. We ought to show that $\mathcal{M} = F_1(M)$, i.e., that $\mathcal{M} \subset F_1(M)$. Indeed, if not, then there is, by the definition of $\mathcal{K}$, a point $b \in K \setminus M$ such that $\{b\} \in \mathcal{M} \setminus F_1(M)$. Define $M'_n = \cup\mathcal{M}_n$ and $M' = \cup\mathcal{M}$. Then $b \in M' \setminus M$ and the sequence of M'_n tends to M' which contains M properly. This contradicts to the fact that M is a maximal limit continuum in K.

Claim 2. There is a maximal limit continuum in $\mathcal{K}$ contained in $C(M) \cap \mu^{-1}(t)$.
For shortness put $\mathcal{B}_n = C(M_n) \cap \mu^{-1}(t)$. By compactness

(taking a convergent subsequence if necessary) we may assume that the sequence $\mathcal{B}_n$ converges to a continuum $\mathcal{B} \subset \mu^{-1}(t) \cap C(M)$. We will show that $\mathcal{B}$ is a maximal limit continuum in $\mathcal{K}$. Take, for each $n \in \mathbb{N}$, continua $\mathcal{B}'_n \supset \mathcal{B}_n$ with $\operatorname{Lim} \mathcal{B}'_n = \mathcal{B}' \supsetneq \mathcal{B}$. Define $M'_n = \cup \mathcal{B}'_n$. Then $M_n \subset M'_n$ and $\operatorname{Lim} M'_n = \cup \mathcal{B}' \subset \cup \mathcal{K} = K$, so $\operatorname{Lim} M'_n = M$ because M is a maximal limit continuum in K. Since $a \notin M$, the union $\cup \mathcal{A}$ is not in $\mathcal{B}'$, while $\mathcal{B}$ (which is a subset of $\mathcal{B}'$) is contained in $C(M) \cap \mu^{-1}(t)$. This implies, by the definition of $\mathcal{K}$, that we have $\mathcal{B}' \subset C(K) \cap \mu^{-1}(t)$. Suppose that $\mathcal{B} \subsetneq \mathcal{B}'$. Then there is $\varepsilon > 0$ such that $\mathcal{H}(\mathcal{B}, \mathcal{B}') > \varepsilon$, whence it follows that $\mathcal{H}(\mathcal{B}_n, \mathcal{B}'_n) > \varepsilon$ for almost all $n \in \mathbb{N}$. Choose a convergent sequence of continua $P_n \in \mathcal{B}'_n$ with $P = \operatorname{Lim} P_n \in \mathcal{B}' \setminus \mathcal{B}$. Since $\mathcal{B}' \subset \mu^{-1}(t)$, we conclude that the sequence $\mu(P_n)$ tends to $\mu(P) = t$. Therefore, by the definition of $\mathcal{B}_n$, we have $H(P_n, C(M_n)) > \varepsilon$ for almost all $n \in \mathbb{N}$. Then the continua P_n are not contained in $N_X(M_n, \varepsilon)$, and consequently P is not contained in $N_X(M, \varepsilon)$, contrary to the fact that $P \subset \cup \mathcal{B}' = M$. The proof is complete.

Corollary 4.6. (Wardle) *Let a continuum X be given. If the hyperspace $C(X)$ has the property of Kelley, then X has the property of Kelley.*

As it is indicated in [18, Questions (16.37), p. 558] it can be deduced from Wardle's proof of Theorem 2.8 in [20, p. 294] (i.e., Corollary 4.6 above) that the same implication holds with 2^X in place of $C(X)$. Analogously to Theorem 4.5 the mentioned result can be sharpened as follows.

Theorem 4.7. *Let a continuum X be given. If the hyperspace 2^X is a semi-Kelley continuum, then X has the property of Kelley.*

Proof: Suppose that X does not have the property of Kelley. Then by Proposition 3.15 there exists a subcontinuum K of X and a (strong) maximal limit continuum M in K such that M is a nondegenerate proper subset of K. Take a sequence of subcontinua M_n of X converging to M as in the definition of a maximal limit continuum in K. Fix a point $a \in K \setminus M$ and a point $b \in X \setminus K$. Let $\mathcal{A}$ be an order arc in 2^X from $\{a\}$ to X. Define

$$\mathcal{K}_b = \{\{x, b\} : x \in K\} \quad \text{and} \quad \mathcal{M}_b = \{\{x, b\} : x \in M\},$$

and let $\mathcal{B}$ be an order arc in 2^X from $\{a, b\}$ to X. Put

$$\mathcal{K} = \mathcal{A} \cup \mathcal{B} \cup F_1(K) \cup \mathcal{K}_b.$$

We will show that $F_1(M)$ and $\mathcal{M}_b$ are maximal limit continua in $\mathcal{K}$. Since they obviously are disjoint, this will finish the proof.

Claim 1. $F_1(M)$ is a maximal limit continuum in $\mathcal{K}$. The argument is very similar to that for Claim 1 in the proof of Theorem 4.5. Indeed, as previously note that $F_1(M_n)$ are continua in 2^X tending to $F_1(M)$. For each $n \in \mathbb{N}$ let $\mathcal{M}_n$ be a subcontinuum of 2^X such that $F_1(M_n) \subset \mathcal{M}_n$ and that the sequence $\{\mathcal{M}_n\}$ converges to a continuum $\mathcal{M}$ in $\mathcal{K}$. Obviously $F_1(M) \subset \mathcal{M}$. We ought to show that $\mathcal{M} = F_1(M)$, i.e., that $\mathcal{M} \subset F_1(M)$. In fact, if not, then there is, by the definition of $\mathcal{K}$, a point $p \in K \setminus M$ such that $\{p\} \in \mathcal{M} \setminus F_1(M)$. Define $M'_n = \cup \mathcal{M}_n$ and $M' = \cup \mathcal{M}$, and note that M'_n and M' are continua according to [18, Lemma 1.43, p. 97]. Then $p \in M' \setminus M$ and the sequence of M'_n tends to M' which contains M properly. This contradicts to the fact that M is a maximal limit continuum in K.

Claim 2. $\mathcal{M}_b$ is a maximal limit continuum in $\mathcal{K}$. Note that the sets $\mathcal{M}_n = \{\{x, b\} : x \in M_n\}$ are continua in 2^X tending to $\mathcal{M}_b$. For each $n \in \mathbb{N}$ let $\mathcal{M}'_n$ be a subcontinuum of 2^X such that $\mathcal{M}_n \subset \mathcal{M}'_n$ and that the sequence $\{\mathcal{M}'_n\}$ is convergent with $\mathcal{M}' = \operatorname{Lim} \mathcal{M}'_n \subset \mathcal{K}$. Obviously $\mathcal{M}_b \subset \mathcal{M}'$. We ought to show that $\mathcal{M}_b = \mathcal{M}'$, i.e., that $\mathcal{M}' \subset \mathcal{M}_b$. Indeed, if not, then there is, by the definition of $\mathcal{K}$, a point $p \in K \setminus M$, (equivalently, $\{p, b\} \in \mathcal{K}_b \setminus \mathcal{M}_b$), such that $\{p, b\} \in \mathcal{M}' \setminus \mathcal{M}_b$. Define $M'_n = (\cup \mathcal{M}'_n) \setminus \{b\}$ and $M' = (\cup \mathcal{M}') \setminus \{b\}$. Then M'_n and M' are subcontinua of X. Further, $p \in M' \setminus M$ and the sequence of M'_n tends to M' which contains M properly.

This contradicts to the fact that M is a strong maximal limit continuum in K. Thus the proof is complete.

Corollary 4.8. (Wardle) *Let a continuum X be given. If the hyperspace 2^X has the property of Kelley, then X has the property of Kelley.*

A space X is said to be *connected im kleinen at a point* $p \in X$ provided that each neighborhood U of p contains a neighborhood V of p such that for each point $q \in V$ there is a connected subset of U containing both p and q. It is shown in [16, Theorem 2.9, p. 228] that if a continuum X has the property of Kelley, then connectedness im kleinen of the hyperspace $C(X)$ at $A \in C(X)$ implies connectedness im kleinen of $C(X)$ at any $B \in C(X)$ such that $A \subset B$. This result can be generalized from continua having the property of Kelley to all semi-Kelley continua.

Theorem 4.9. *Let a continuum X be semi-Kelley. If the hyperspace $C(X)$ is connected im kleinen at $A \in C(X)$, and if $A \subset B \in C(X)$, then $C(X)$ is connected im kleinen at B.*

Proof: Suppose on the contrary that there exist subcontinua A and B of a semi-Kelley continuum X such that $A \subsetneq B$ and that

(4.10) $C(X)$ is connected im kleinen at A,
(4.11) $C(X)$ is not connected im kleinen at B.

By (4.11) there exists an open set U of X such that $B \subset U$ and a sequence of subcontinua $B_n \subset U$ which converges to B and has the property that B and B_n are in different components of U (see [18, Theorem 1.143, p. 156]). Let $A' \subset A$ be a continuum such that

(4.12) there exists a sequence of continua A_k such that $A_k \subset B_{n_k}$ for some subsequence $\{n_k\}$ of indices n, with $A' = \operatorname{Lim} A_k$,
(4.13) A' is the nearest to A subcontinuum of A satisfying condition (4.12).

The latter condition means that the distance $H(A', A)$ is the minimum among the distances $H(P, A)$ for all $P \in C(A)$ such that $P = \operatorname{Lim} A_k$, where the sequence $\{A_k\}$ is as in (4.12). The existence of such A' is a consequence of compactness of the family of all continua satisfying (4.12). By its construction, A' is a strong maximal limit continuum in A.

Note that (4.12) implies that $C(X)$ is not connected im kleinen at A', whence it follows that $A \setminus A' \neq \emptyset$.

Pick up a point

(4.14) $a \in A \setminus A'$,

and choose a sequence of points $a_n \in B_n$ with $a = \lim a_n$. Let A'' be a maximal (with respect to inclusion) element of the family $\mathcal{M}(\{\{a_n\}\}) \cap C(A)$. Then, by Proposition 3.9, A'' is a strong maximal limit continuum in A. By (4.14) we have $a \in A'' \setminus A'$. Since X is a semi-Kelley continuum, $A' \subset A''$. Then the continuum A'' is nearer to A than A', a contradiction with (4.13). The argument is complete.

Corollary 4.15. (Moon, Hur and Rhee) *Let a continuum X have the property of Kelley. If the hyperspace $C(X)$ is connected im kleinen at $A \in C(X)$, and if $A \subset B \in C(X)$, then $C(X)$ is connected im kleinen at B.*

An example below shows that being a semi-Kelley continuum is an essential assumption in this result.

Example 4.11. *There is a continuum X which is not semi-Kelley and which contains two subcontinua A and B with $A \subset B$ such that $C(X)$ is connected im kleinen at A while it is not at B.*

Proof: Keeping notation of points as in the proof of Example 3.5 put additionally $a_n = (0, 1 + 1/n)$ and define

$$X = va \cup vb \cup vc \cup \bigcup \{bp_n \cup p_n a_n \cup a_n q_n \cup q_n c_n : n \in \mathbb{N}\}.$$

Put $A = va' \cup vb' \cup vc'$ and observe that $va' \cup vb'$ and $va' \cup vc'$ are two maximal limit continua in A neither of which is contained in the other, so X is not semi-Kelley. Note further that $C(X)$ is connected im kleinen at A while not at $B = va \cup vb' \cup vc'$.

5. Semi-Kelley continua and mappings

A mapping $f : X \to Y$ between continua is said to be:
— a *retraction* provided that $Y \subset X$ and the restriction $f|Y$ is the identity on Y; (then Y is called a *retract of* X);
— *open* provided that for each open subset of X its image under f is an open subset of Y;
— *monotone* provided that the point-inverse $f^{-1}(y)$ is connected for each point $y \in Y$;
— *confluent* provided that for each subcontinuum Q of Y each component of the the inverse image $f^{-1}(Q)$ is mapped onto Q under f;
— *weakly confluent* provided that for each subcontinuum Q of Y there is a component of the the inverse image $f^{-1}(Q)$ which is mapped onto Q under f;
— *semi-confluent* provided that for each subcontinuum Q of Y and for every two components C_1 and C_2 of the the inverse image $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$;
— *joining* provided that for each subcontinuum Q of Y and for every two components C_1 and C_2 of the the inverse image $f^{-1}(Q)$ we have $f(C_1) \cap f(C_2) \neq \emptyset$.

The property of Kelley is preserved under retractions, [20, Theorem 2.9, p. 294]. A similar result is true for semi-Kelley continua. To show it we prove the following lemma first.

Lemma 5.1. *Let X, Y and K be continua such that $K \subset Y \subset X$, and that Y is a retract of X. If $M \subset K$ is a (strong) maximal limit continuum in K when K is considered as a subcontinuum of Y, then M is a (strong) maximal limit continuum in K when K is considered as a subcontinuum of X.*

Proof: We will argue for the version with strong maximal limit continua. The argument for the other version (with maximal limit continua) is the same: we need only to take the whole sequence of all positive integers as the subsequence $\{n_k\}$ considered in the proof presented.

Let $f : X \to Y \subset X$ be a retraction. Take a sequence of subcontinua M_n of Y converging to M as in the definition of a strong maximal limit continuum in K. Consider a subsequence $\{M_{n_k}\}$ of the sequence $\{M_n\}$ and a convergent sequence of subcontinua M'_k with $M_{n_k} \subset M'_k$ for each $k \in \mathbb{N}$ and such that $\mathrm{Lim}\, M'_k \subset K$. Then $f(\mathrm{Lim}\, M'_k) \subset f(K) = K$, so $f(\mathrm{Lim}\, M'_k) = \mathrm{Lim}\, M'_k$. On the other hand $f(\mathrm{Lim}\, M'_k) = \mathrm{Lim}\, f(M'_k)$. Since $f(M'_k) \supset f(M_{n_k}) = M_{n_k}$ for each $k \in \mathbb{N}$, we infer from the definition of a strong maximal limit continuum in K that $\mathrm{Lim}\, M'_k = \mathrm{Lim}\, f(M'_k) = M$. The proof is complete.

As a consequence of Lemma 5.1 we get the mentioned result.

Theorem 5.2. *Let X be a continuum and let $f : X \to Y \subset X$ be a retraction. If X is a semi-Kelley continuum, then Y is semi-Kelley as well.*

To prove the next two theorems we need another lemma.

Lemma 5.3. *Let a continuum X have the property of Kelley, and let $f : X \to Y$ be a weakly confluent surjection. If A is a maximal limit continuum in a subcontinuum K of Y, then there exists a component C of $f^{-1}(K)$ such that $f(C) = A$.*

Proof: Let a sequence of continua A_n be given as in the definition of the maximal limit continuum A in K. Since f is weakly confluent, for each $n \in \mathbb{N}$ there is a component B_n of $f^{-1}(A_n)$ such that $f(B_n) = A_n$. Taking a subsequence if necessary we may assume that the sequence $\{B_n\}$ is convergent. Put $B = \mathrm{Lim}\, B_n$. Then $f(B) = f(\mathrm{Lim}\, B_n) = \mathrm{Lim}\, A_n = A$. Let C be the component of $f^{-1}(K)$ that contains B. Choose a convergent sequence of points $b_n \in B_n$ and put $b = \lim b_n$. Since X has the property of Kelley, for each $n \in \mathbb{N}$ there are continua C_n in X containing the points b_n, with $C = \mathrm{Lim}\, C_n$. Then $A_n \cup f(C_n)$ are continua containing A_n. By the definition of a maximal limit continuum we have $\mathrm{Lim}\, (A_n \cup f(C_n)) = A$, so $f(C) = \operatorname{Lim} f(C_n) \subset A$. On the other hand $A = \operatorname{Lim} A_n = \operatorname{Lim} f(B_n) \subset \operatorname{Lim} f(C_n) = f(C)$. Thus $A = f(C)$ as required.

As a consequence we get the following known result, originally due to R. W. Wardle, see [20, Theorem 4.3, p. 296].

Theorem 5.4. (Wardle) *If a continuum X has the property of Kelley and if $f : X \to Y$ is a confluent surjection, then Y has the property of Kelley as well.*

Proof: Let K be a subcontinuum of X and let A be a maximal limit continuum in K. By Lemma 5.3 there exists a component C of $f^{-1}(K)$ such that $f(C) = A$. Since f is confluent, we have $f(C) = K$, whence $K = A$, and therefore Y has the property of Kelley according to the characterization in Theorem 3.11.

Theorem 5.5. *If a continuum X has the property of Kelley, and Y is a semi-confluent image of X, then Y is semi-Kelley.*

Proof: Assume a mapping $f : X \to Y$ is a semi-confluent surjection. Let A and B be maximal limit continua in some subcontinuum K of Y. By Lemma 5.3 there are components C_A and C_B of $f^{-1}(K)$ such that $f(C_A) = A$ and $f(C_B) = B$. Since f is semi-confluent we have either $A \subset B$ or $B \subset A$ as needed.

Semi-confluence of the mapping is an indispensable assumption in Theorem 5.5 and the result cannot be extended to weakly confluent mappings. The next example shows this.

Example 5.6. *There exists a weakly confluent surjective mapping between fans X and Y such that X has the property of Kelley, while Y is not semi-Kelley.*

Proof: Recall that given two points p and q in the plane, we denote by pq the straight line segment with end points p and q. In the polar coordinates (ρ, ϕ) in the plane with pole

$v = (0,0)$, consider for each $n \in \mathbb{N}$ points $d_n = (1, \pi/(2n))$, put $d = (1,0) = \lim d_n$, and define

$$X = vd \cup \bigcup \{vd_n : n \in \mathbb{N}\}.$$

Thus X is the harmonic fan, so it has the property of Kelley. To define Y consider points

$p_n = (\frac{1}{2}, \frac{\pi}{2n}),\ \ q_n = (\frac{1}{4}, \frac{\pi}{2n+1}),\ \ r_n = (1, \frac{\pi}{2n+1}),\ \ s_n = (\frac{3}{4}, \frac{\pi}{2n+2}),$

put $r = (1,0) = \lim r_n$, and define

$$Y = vr \cup \bigcup \{vp_n \cup p_n q_n \cup q_n r_n \cup r_n s_n : n \in \mathbb{N}\}.$$

Thus Y is a fan. Putting $p = (\frac{1}{2}, 0) = \lim p_n$ and $s = (\frac{3}{4}, 0) = \lim s_n$ we see that the singletons $\{p\}$ and $\{s\}$ are strong maximal limit continua in the continuum $ps \subset vr$, and thereby Y is not semi-Kelley, according to (3.20) of Theorem 3.18.

To define a weakly confluent surjection $f : X \to Y$ consider for each $n \in \mathbb{N}$ in the straight line segment $vd_n \subset X$ points $a_n = (\frac{1}{4}, \frac{\pi}{2n})$, $b_n = (\frac{1}{2}, \frac{\pi}{2n})$, $c_n = (\frac{3}{4}, \frac{\pi}{2n})$, and put

$$a = (\frac{1}{4}, 0) = \lim a_n, \quad b = (\frac{1}{2}, 0) = \lim b_n, \quad c = (\frac{3}{4}, 0) = \lim c_n.$$

Thus a, b and c lie in $vd \subset X$. Let f be a piecewise linear mapping determined by putting $f(v) = v$ and, for each $n \in \mathbb{N}$,

$$f(a_n) = p_n, \quad f(b_n) = q_n, \quad f(c_n) = r_n, \quad f(d_n) = s_n,$$

and consequently $f(a) = p$, $f(b) = q = (\frac{1}{4}, 0) = \lim q_n$, $f(c) = r$ and $f(d) = s$. The partial mappings $f|va_n : va_n \to vp_n$, $f|a_n b_n : a_n b_n \to p_n q_n$, $f|b_n c_n : b_n c_n \to q_n r_n$ and $f|c_n d_n : c_n d_n \to r_n s_n$ are linear. Therefore the partial mappings $f|va : va \to vp$, $f|ab : ab \to pq$, $f|bc : bc \to qr$ and $f|cd : cd \to rs$ also are linear. Observe that, for each $n \in \mathbb{N}$, the mapping f maps the segment vd_n homeomorphically onto the broken line $vp_n \cup p_n q_n \cup q_n r_n \cup r_n s_n$, and that the segment vd is mapped onto the segment vr. Since each mapping onto an arc is weakly confluent, [19, Lemma, p. 236], it can easily be deduced that $f : X \to Y$ is weakly confluent. The argument is then complete.

Our next example shows that the conclusion of Theorem 5.5 is not true if a larger class of joining mappings is considered in place of semi-confluent ones.

Example 5.7. *There are fans X and Y and a joining surjective mapping $f : X \to Y$ such that X has the property of Kelley and Y is not semi-Kelley.*

Proof: Keeping notation of points v, b, c, b_n and c_n as in Example 3.5, put

$$X = vb \cup vc \cup \bigcup \{vb_n \cup vc_n : n \in \mathbb{N}\}.$$

Then X is homeomorphic to the one-point union of two harmonic fans having their tops in common only. Therefore it has the property of Kelley. Define an equivalence relation $\sim$ on X in such a way that for two distinct points $p = (x, y)$ and $p' = (x', y')$ we have $p \sim p'$ if and only if $y = y' = 0$, $xx' \geq 0$ and $|x + x'| = 1$. Let $Y = X/\sim$ be the quotient space, and $f : X \to Y$ be the quotient mapping. Thus f identifies two points if and only if either they are both in vb and are symmetric with respect to the mid point $b' = (-1/2, 0)$ of vb, or they are both in vc and are symmetric with respect to the mid point $c' = (1/2, 0)$ of vc. Therefore the restriction $f|(X \setminus (vb \cup vc))$ is a homeomorphism. Considering various subcontinua of Y one can verify by the definition that f is joining. To check that Y is not semi-Kelley take points $b'' = (-1/4, 0)$ and $c'' = (1/4, 0)$ in X and define $K \subset Y$ as the arc with end points $f(b'')$ and $f(c'')$. Then its subarcs $f(vb'')$ and $f(vc'')$ are maximal limit continua in K having $f(v)$ as the only point in common. The argument is complete.

Unlike the property of Kelley (see Theorem 5.4 above) the property of being semi-Kelley is not preserved under confluent mappings. The next example shows this.

Example 5.8. *There are continua X and Y and a confluent surjective mapping $f : X \to Y$ such that X is semi-Kelley, while Y is not.*

Proof: In the Euclidean plane let C_1 be the cone with the vertex $(1,1)$ over the set $\{(1+1/n, 0) : n \in \mathbb{N}\}$ and let C_2 be the cone with the vertex $(-1,-1)$ over the set $\{(-1-1/n, 0) : n \in \mathbb{N}\}$. Put

$$X = (\{-1,1\} \times [-1,1]) \cup ([-1,1] \times \{-1,1\}) \cup C_1 \cup C_2.$$

Then X is a semi-Kelley continuum. Next identify in X the points (x,y) and (x',y') if and only if $y = y'$, $|x| \leq 1$ and $|x'| \leq 1$. In other words, the identification shrinks in X each of the two segments $[-1,1] \times \{-1\}$ and $[-1,1] \times \{1\}$ to a point, and glue together the points $(-1,y)$ and $(1,y)$ for $y \in [-1,1]$. Let $f : X \to Y$ be the identification mapping. Then the resulting continuum Y is homeomorphic to the union of the cone with the vertex $(0,1)$ over the set $\{(0,0)\} \cup \{(1/n, 0) : n \in \mathbb{N}\}$ and its image under central symmetry, i.e., the cone with the vertex $(0,-1)$ over the set $\{(0,0)\} \cup \{(-1/n, 0) : n \in \mathbb{N}\}$. The reader can verify that Y is not semi-Kelley.

In the light of Theorem 5.2 and Example 5.8 the following questions are very natural.

Question 5.9. What classes of mappings preserve the property of being semi-Kelley? In particular, is the property preserved under (a) monotone, (b) open mappings?

It is known that if a continuum Y is locally connected and X is an arbitrary continuum, then the uniform limit of monotone mappings from X onto Y is monotone (see [21, Corollary 3.11, p. 174] and [22, Theorem, p. 466]; compare also [13, Theorem 1, p. 797], where a generalization to compact Hausdorff spaces is presented). Local connectedness of the range space is an essential assumption, because of Whyburn's example of a near homeomorphism of the harmonic fan onto itself which is not monotone, see the example in [22, p. 465]. Recall that a *near homeomorphism* is defined as the uniform limit of homeomorphisms. The example shows also that local connectedness cannot be relaxed to having the property of Kelley.

Nadler has shown in [17, Theorem 3.1, p. 570, and Corollary 3.3, p. 571] that if a continuum Y has the property of Kelley and X is an arbitrary continuum, then (a) the space of all confluent mappings $f : X \to Y$ is a closed subspace of the space Y^X of all mappings from X to Y, and (b) each near homeomorphism from X onto Y is confluent. One can ask if the assumption on Y in either (a) or (b) can be relaxed to being semi-Kelley. The answer to both these questions is negative, because of an example constructed by Nadler in [17, Example (1.4), p. 564]. The idea of the example is patterned after Whyburn's example mentioned above. We repeat the example here for further applications.

Example 5.10. *Let Y be the harmonic prolonged fan (which is a semi-Kelley continuum). There exists a near homeomorphism $f : Y \to Y$ which is not confluent.*

Proof: In the Euclidean plane let pq denote the straight line segment joining p and q. Put $v = (0, 1)$, $e = (0, 0)$ and $e_n = (1/n, 0)$ for each $n \in \mathbb{N}$, and define $X = ve \cup \bigcup\{ve_n : n \in \mathbb{N}\}$. Then X is the harmonic fan. Further, for each $n \in \mathbb{N}$, let a_n be the midpoint of the segment ve_n, and put $Y = ve \cup \bigcup\{va_n : n \in \mathbb{N}\}$ be the continuum defined there. Y does not have the property of Kelley, but it is semi-Kelley. For each $n \in \mathbb{N}$ take a homeomorphism $f_n : Y \to Y$ described as follows. $f_n|va_1 : va_1 \to va_n$ is a linear mapping with $f_n(v) = v$; for $m \in \{2, \dots, n\}$ let $f_n|va_m : va_m \to va_{m-1}$ be linear; for $m > n$ the restriction $f_n|va_m : va_m \to va_m$ is the identity, and finally $f_n|ve : ve \to ve$ is the identity as well. Then the limit mapping $f = \lim f_n$ maps va_1 linearly onto va, where $a = (0, 1/2)$ is the midpoint of the segment ve, for each $m > 1$ the restrictions $f|va_m : va_m \to va_{m-1}$ are linear, and $f|ve : ve \to ve$ is the identity. Since $f^{-1}(ae)$ has a one-point component, hence f is not confluent.

The next two results are related to the above mentioned theorems of Nadler, [17, Theorem 3.1, p. 570, and Corollary 3.3, p. 571].

Proposition 5.11. *Let a mapping $f : X \to Y$ between continua be the uniform limit of confluent mappings. Then for each subcontinuum K of Y and for each component C of $f^{-1}(K)$ the image $f(C)$ is a strong maximal limit continuum in K.*

Proof: For each $n \in \mathbb{N}$ let $f_n : X \to Y$ be a confluent mapping, and let $f = \lim f_n$ be the uniform limit. We will show that putting $M_n = f_n(C)$ the conditions of Definition 3.3 are satisfied. Note that $f(C) = \operatorname{Lim} M_n$. Consider a subsequence $\{M_{n_k}\}$ of the sequence $\{M_n\}$, and for each $k \in \mathbb{N}$ let M'_k be a continuum such that $M_{n_k} \subset M'_k$. Assume that the sequence $\{M'_k\}$ is convergent, and let $M' = \operatorname{Lim} M'_k \subset K$. We have to show that $M' = f(C)$. Since $f_{n_k}(C) \subset M'_k$, it follows that $C \subset f_{n_k}^{-1}(M'_k)$, hence there is a component C_k of $f_{n_k}^{-1}(M'_k)$ that contains C. Taking a convergent sequence if necessary, we may assume that the sequence $\{C_k\}$ tends to a continuum C_0. Obviously $C \subset C_0$. By confluence of the mappings f_n we get $f_{n_k}(C_k) = M'_k$. Now if k tends to infinity we get (for the limits of both mappings and continua) $f(C_0) = \operatorname{Lim} M'_k = M' \subset K$. Since $C \subset C_0$ and C is a component of $f^{-1}(K)$, the equality $C = C_0$ follows, and thereby $f(C_0) = M' = f(C)$. The proof is complete.

Corollary 5.12. *If a continuum Y is semi-Kelley, X is an arbitrary continuum, and a surjection $f : X \to Y$ is the uniform limit of confluent mappings, then f is semi-confluent.*

Proof: Let K be a subcontinuum of Y, and let C_1 and C_2 be components of $f^{-1}(K)$. Then $f(C_1)$ and $f(C_2)$ are strong maximal limit continua in K by Proposition 5.11, and since the continuum Y is semi-Kelley, one of them is contained in the other, by (3.20) of Theorem 3.18. So f is semi-confluent, as needed.

The assumption that the mapping f in Corollary 5.12 is the uniform limit of confluent mappings cannot be relaxed to being the uniform limit of semi-confluent ones. In other words, if a continuum Y is semi-Kelley and X is an arbitrary continuum, then the space of all semi-confluent mappings $f : X \to Y$ need not be a closed subspace of the space Y^X of all mappings from X to Y. The next example shows this.

Example 5.13. *There exists a semi-Kelley continuum Z and a mapping $f : Z \to Z$ which is not semi-confluent, being however the uniform limit of semi-confluent mappings.*

Proof: We keep notation of Example 5.10. For each $n \in \mathbb{N}$, let b_n, c_n and d_n be the midpoints of the segments va_n, a_ne_n, and c_nc_{n+1}, respectively. Note that the second (y) coordinates of points d_n, c_n, a_n and b_n are 1/4, 1/4, 1/2 and 3/4, respectively. Put $c = \lim c_n = \lim d_n$, $a = \lim a_n$, $b = \lim b_n$. Then a, b and c are midpoints of the segments ve, va and ae, correspondingly. Define $Z = ve \cup \bigcup\{ve_n \cup e_nd_n : n \in \mathbb{N}\}$. Then Z is homeomorphic to the harmonic hooked fan (2.3), so it is semi-Kelley by construction.

For each $n \in \mathbb{N}$ let $f_n : Z \to Z$ be defined as follows. The restriction $f_n|(ve_1 \cup e_1d_1) : (ve_1 \cup e_1d_1) \to (ve_n \cup e_nd_n)$ is a piecewise linear mapping with:

$$f_n(v) = v,\ f_n(b_1) = a_n,\ f_n(a_1) = b_n,$$
$$f_n(c_1) = c_n,\ f_n(e_1) = e_n,\ f_n(d_1) = d_n,$$

such that the restrictions $f_n|vb_1$, $f_n|b_1a_1$, $f_n|a_1c_1$, $f_n|c_1e_1$ and $f_n|e_1d_1$ are linear. For $m \in \{2, \dots, n\}$ let $f_n|(ve_m \cup e_md_m) : (ve_m \cup e_md_m) \to (ve_{m-1} \cup e_{m-1}d_{m-1})$ be a linear homeomorphism with $f_n(e_m) = e_{m-1}$ and $f_n(d_m) = d_{m-1}$, such that the restrictions $f_n|ve_m$ and $f_n|e_md_m$ are linear. Finally define $f_n|(ve \cup \bigcup\{ve_m \cup e_md_m : m \in \mathbb{N} \text{ and } m > n\})$ as the identity. It is evident that for each $n \in \mathbb{N}$ the mapping f_n just defined is semi-confluent. Put $f = \lim f_n$ and observe that:
1) f maps $ve_1 \cup e_1d_1$ onto ve piecewise linearly with $f(v) = v$, $f(b_1) = a$, $f(a_1) = b$, $f(c_1) = c$, $f(e_1) = e$, $f(d_1) = c$, so that the restrictions $f|vb_1$, $f|b_1a_1$, $f|a_1c_1$, $f|c_1e_1$ and $f|e_1d_1$ are linear;

2) for each $m > 1$ the restriction $f|(ve_m \cup e_m d_m) : (ve_m \cup e_m d_m) \to (ve_{m-1} \cup e_{m-1} d_{m-1})$ is a linear homeomorphism with $f(e_m) = e_{m-1}$ and $f(d_m) = d_{m-1}$ such that $f|ve_m$ and $f|e_m d_m$ are linear mappings;
3) $f|ve$ is the identity.

Take the segment ac as a subcontinuum of the range space, and note that $f^{-1}(ac)$ has four components: ac, $\{b_1\}$, $\{d_1\}$, and the fourth one, containing the point c_1 and contained in a_1c_1. Since the degenerate components have distinct images, namely the singletons $\{a\}$ and $\{c\}$, the mapping f is not semi-confluent. The proof is finished.

Remark 5.14. T. Maćkowiak has shown in [14, Theorem 2, p. 71] that if the continuum Y is locally connected, then the uniform limit of semi-confluent mappings from X onto Y is semi-confluent. Local connectedness of Y is essential in this result, see [15, Example 5.62, p. 47]. Our Example 5.13 shows that local connectedness of Y cannot be relaxed to being semi-Kelley. Therefore, in connection with Corollary 5.12 and Example 5.13 the following question is interesting and natural.

Question 5.15. Is it true that if a continuum Y has the property of Kelley and X is an arbitrary continuum, then the uniform limit of semi-confluent mappings from X onto Y is semi-confluent?

Recall that a continuum Z is said to be *contractible* provided that the identity mapping from Z onto Z is homotopic to a constant mapping from Z into Z. It is known (see [12, Theorem 3.3, p. 26]; compare [18, Theorem 16.15, p. 544]) that if X has the property of Kelley, then the hyperspaces 2^X and $C(X)$ are contractible. It is interesting to know whether the assumption concerning the continuum X can be relaxed.

Question 5.16. Is it true that if a continuum X is semi-Kelley, then the hyperspace 2^X and/or $C(X)$ is contractible?

References

1. J. J. Charatonik, *Confluent mappings and unicoherence of continua*, Fund. Math., **56** (1964), 213–220.
2. J. J. Charatonik, *The property of Kelley and confluent mappings*, Bull. Polish Acad. Sci. Math., **31** (1983), 375–380.
3. J. J. Charatonik and W. J. Charatonik, *Fans with the property of Kelley*, Topology Appl., **29** (1988), 73–78.
4. J. J. Charatonik and W. J. Charatonik, *The property of Kelley for fans*, Bull. Polish Acad. Sci. Math., **36** (1988), 169–173.
5. J. J. Charatonik, W. J. Charatonik and S. Miklos, *Confluent mappings of fans*, Dissertationes Math. (Rozprawy Mat.), **301** (1990), 1–86.
6. W. J. Charatonik, *Hyperspaces and the property of Kelley*, Bull. Acad. Polon. Sci. Ser. Sci. Math., **30** (1982), 457–459.
7. W. J. Charatonik, *On the property of Kelley in hyperspaces*, Lecture Notes in Math., Topology, Proc. International Topological Conference, Leningrad 1982, vol. 1060, Springer Verlag, 1984, 7–10.
8. W. J. Charatonik, *A homogeneous continuum without the property of Kelley*, Topology Appl. (to appear).
9. H. Kato, *Generalized homogeneity of continua and a question of J. J. Charatonik*, Houston J. Math., **13** (1987), 51–63.
10. H. Kato, *On the property of Kelley in the hyperspace and Whitney continua*, Topology Appl., **30** (1988), 165–174.
11. H. Kato, *A note on continuous mappings and the property of J. L. Kelley*, Proc. Amer. Math. Soc., **112**, (1991), 1143–1148.
12. J. L. Kelley, *Hyperspaces of a continuum*, Trans. Amer. Math. Soc., **52** (1942), 22–36.
13. K. Kuratowski and R. C. Lacher, *A theorem on the space of monotone mappings*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., **17** (1969), 797–800.
14. T. Maćkowiak, *On sets of confluent and related mappings in the space Y^X*, Colloq. Math., **36** (1976), 69–80.
15. T. Maćkowiak, *Continuous mappings on continua*, Dissertationes Math. (Rozprawy Mat.), **158** (1979), 1–91.
16. J. R. Moon, K. Hur and C. J. Rhee, *Connectedness im kleinen and components in $C(X)$*, Bull. Korean Math. Soc., **34** (1997), 225–231.
17. S. B. Nadler, Jr., *Concerning completeness of the space of confluent mappings*, Houston J. Math., **2** (1976), 561–580.
18. S. B. Nadler, Jr., *Hyperspaces of sets*, M. Dekker, 1978.
19. D. R. Read, *Confluent and related mappings*, Colloq. Math., **29** (1974), 233–239.

20. R. W. Wardle, *On a property of J. L. Kelley*, Houston J. Math., **3** (1977), 291–299.
21. G. T. Whyburn, Analytic topology, Amer. Math. Soc. Colloq. Publ. 28, Providence, 1942, reprinted with corrections 1971.
22. G. T. Whyburn, *Monotoneity of limit mappings*, Duke Math. J., **29** (1962), 465–470.

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