Abstract—The discrete-time bounded-real lemma for nonminimal discrete systems is presented. Based on this lemma, rigorous necessary and sufficient conditions for the existence of positive definite solutions to the Lyapunov equation for \( n \)-dimensional (\( n \)-D) digital systems are proposed. These new conditions can be applied to \( n \)-D digital systems with \( n \)-D characteristic polynomials involving factor polynomials of any dimension, 1-D to \( n \)-D. Further, the results in this paper show that the positive definite solutions to the \( n \)-D Lyapunov equation of an \( n \)-D system with characteristic polynomial involving 1-D factors can be obtained from the solutions of a \( k \)-D system \((0 \leq k \leq n)\) subsystem and \( m \) (\(1 \leq m \leq n\)) 1-D subsystems. This could significantly simplify the complexity of solving the \( n \)-D Lyapunov equation for such cases.

Index Terms—Lyapunov equation, multidimensional systems, stability.

I. INTRODUCTION

During the last two decades, many authors have studied stability and the Lyapunov equation for \( n \)-dimensional (\( n \)-D) systems (see [1]–[10] and the references therein). The stability of \( n \)-D systems is essential for the design and implementation of such systems and the \( n \)-D Lyapunov equation has many useful applications for \( n \)-D systems in state-space description [3], [4], [8], [10].

The relationship between stability and the Lyapunov equation for \( n \)-dimensional (\( n \)-D) digital systems described by a state-space model is very important for the stability analysis, design and finite wordlength implementation of such systems. It is well known that the existence of positive definite matrices satisfying the \( n \)-D Lyapunov equation is sufficient for stability of \( n \)-D systems but not necessary, as it was shown with an example in [6] for the 2-D case. Further, in [6] it was shown that necessary and sufficient conditions for the existence of positive definite solutions for the 2-D Lyapunov equation can be obtained from the discrete-time strictly bounded real lemma. This lemma formulates the strictly bounded real property of a minimal realization of a transfer matrix and leads, therefore, to conditions involving the requirement of reachability and observability of \( (A_{22}, A_{21}) \) and \( (A_{12}, A_{11}) \) where

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}
\]  

(1)

is the system matrix of the 2-D system. Similar results were also established for \( n \)-D (\( n \geq 3 \)) systems in [7]. If the reachability and/or observability conditions are not satisfied, the results of [6] and [7] cannot be applied directly. For the 2-D case, loss of reachability and/or observability means that the characteristic polynomial of the 2-D system contains one (or two) 1-D polynomials. It was shown in [6] that such a case the original 2-D system can be decomposed into two subsystems; a 1-D subsystem and a 2-D subsystem which satisfies both the reachability and observability conditions. Then the theorem in [6] can be used for the Lyapunov equation of the 2-D subsystem. The question whether a positive definite solution to the Lyapunov equation of the original 2-D system does exist and how it can be found was left open in [6]. It was only mentioned that such a solution may not necessarily exist even if positive definite solutions of the Lyapunov equations for both the 1-D and the 2-D subsystems exist.

2-D systems described with a system matrix \( A \), (1), with the reachability and/or observability conditions not satisfied, have characteristic polynomials involving 1-D factors, as mentioned above. It is interesting to note that this is a more general case than the so-called separable denominator 2-D systems, i.e., systems with a characteristic polynomial which is a product of two 1-D polynomials. In state space description, such systems are usually defined as systems with a system matrix \( A \) where either \( A_{21} \) or \( A_{12} \) is zero [12]–[14]. It is easy to see that this is only a sufficient but not necessary condition. The necessary and sufficient condition for the 2-D characteristic polynomial of \( A \) to be a product of two 1-D polynomials is that all the reachable parts in \( (A_{22}, A_{21}) \) or \( (A_{12}, A_{11}) \) are unobservable in \( (A_{12}, A_{11}) \) or \( (A_{11}, A_{12}) \). This follows easily from the discussions in [6] and [11]. If some of the reachable parts in \( (A_{22}, A_{21}) \) or \( (A_{12}, A_{11}) \) are observable in \( (A_{22}, A_{12}) \) or \( (A_{11}, A_{21}) \), then the 2-D characteristic polynomial of \( A \) is a product of a 2-D and a 1-D polynomials. The relationship between these polynomials and the system matrix \( A \) is discussed in [6]. These observations for 2-D systems can be extended to \( n \)-D systems having characteristic polynomials which are products of \( k \)-D polynomials with \( (0 \leq k \leq n) \).
that many symmetries in the frequency response specifications lead to \( n \)-D digital filter designs with \( n \)-D characteristic polynomials which include \( k \)-D \((0 \leq k \leq n)\) factors. See [15] for discussions and further references.

In this paper the strictly bounded real lemma for nonminimal realizations of a transfer matrix is presented and this lemma is used to formulate new necessary and sufficient conditions for the existence of positive definite solutions to the \( n \)-D Lyapunov equation. These conditions do not require reachability and observability and thus can be applied to a much larger class of \( n \)-D (including 2-D) digital systems than earlier conditions in [6] and [7]. They can be applied to \( n \)-D systems which have characteristic polynomials which involve 1-D polynomial factors. It is shown in this paper that such systems can be decomposed into one \( k \)-D \((0 \leq k \leq n)\) subsystem and \( m \) \((1 \leq m \leq n)\) 1-D subsystems. Further, the new conditions can be used to show that if positive definite solutions for the 1-D and \( k \)-D subsystems exist, then positive definite solutions for the original system will also exist. An algorithm for finding such solutions based on the positive definite solutions of the subsystems is also presented, indicating that the proposed technique simplifies the solution of the \( n \)-D Lyapunov equation for such cases.

The paper is organized as follows. In Section II the discrete-time strictly bounded real lemma (DTSBRL) for nonminimal realizations of 1-D digital transfer functions is developed. In Section III the new necessary and sufficient condition for the existence of positive definite solutions to the \( n \)-D Lyapunov equation are presented. This new condition is based on the formulation of the DTSBRL in Section II for nonminimal realizations and the implications of these new conditions are discussed. In Section IV an example is considered to illustrate the theoretical results.

II. THE DTSBRL FOR NONMINIMAL SYSTEMS

Consider a square real rational transfer matrix \( S(z^{-1}) \) of a \( k \)-input and \( k \)-output 1-D digital system. Strictly bounded realness is defined as follows [6], [9]:

**Definition 1:** Let \( S(z^{-1}) \) be a square real rational transfer matrix. Then \( S(z^{-1}) \) is called strictly bounded real if and only if

1) all poles of each entry of \( S(z^{-1}) \) lie in \( |z| < 1 \)
2) \( I - S(e^{j\omega})S(e^{-j\omega}) \) is positive definite for all \( \omega \in [0, 2\pi] \).

Let the quadruple \( \{F, G, H, J\} \) be a general (nonminimal) realization of \( S(z^{-1}) \) such that

\[
S(z^{-1}) = J + H(z^{-1})I - F)^{-1}G
\]  

then the discrete-time strictly bounded real lemma can be given:

**Lemma 1:** Let \( S(z^{-1}) \) be a square real rational transfer matrix and \( \{F, G, H, J\} \) a realization of \( S(z^{-1}) \) with the spectral radius of \( F \) being smaller than unit, i.e., \( \rho(F) < 1 \). Then \( S(z^{-1}) \) is strictly bounded real if and only if there exists a symmetric positive definite matrix \( P \) such that the matrix \( Q_1 \) given by

\[
Q_1 = \begin{bmatrix}
I - JTJ - GTPG & -(F^TPG + HJ)^T \\
-F^TPG & \theta
\end{bmatrix}
\]  

is positive definite.

**Remark 1:** This lemma is an extended version of Lemma 1 in [6] for general realizations including minimal and non-minimal realizations of \( S(z^{-1}) \). The sufficient part of this lemma for nonminimal realizations was stated by a remark [6] without proof. The proof of both the sufficient and the necessary part of this lemma are rather lengthy and are included in the Appendix.

III. THE \( n \)-D LYAPUNOV CONDITION

Let us now propose the \( n \)-D Lyapunov condition and compare it with the condition for \( n \)-D stability based on the strictly bounded real lemma presented in the previous section.

Consider a linear shift invariant \( n \)-D digital system described by Roesser’s state-space model of the following form [7]:

\[
x_1(i_1 + 1, i_2, \ldots, i_n)
\]

... 

\[
 x_n(i_1, i_2, \ldots, i_n + 1) = A_{11} x_1(i_1, i_2, \ldots, i_n) + A_{12} x_2(i_1, i_2, \ldots, i_n) + \cdots + A_{1n} x_n(i_1, i_2, \ldots, i_n)
\]

\[
 + B_1 u(i_1, i_2, \ldots, i_n)
\]

\[
 \equiv A X(i) + B u(i)
\]  

\[
y(i_1, i_2, \ldots, i_n) = C_1 x_1(i_1, i_2, \ldots, i_n) + D_1 u(i_1, i_2, \ldots, i_n)
\]

\[
 \equiv C X(i) + D u(i)
\]

where \( x_j(i) \in \mathbb{R}^{n_j \times 1} \) are the current state variables in the \( j \)-th block of \( X(i) \), \( u(i) \) and \( y(i) \) are the inputs and outputs, respectively, with \( A_{ij} \in \mathbb{R}^{n_i \times n_j} \), \( B_i \in \mathbb{R}^{n_i \times s} \), \( C_j \in \mathbb{R}^{p \times n_j} \), \( D \in \mathbb{R}^{p \times s} \).

The stability of an \( n \)-D system described by (4) and (5) depends on the locations of the zeros of the characteristic polynomial.

\[
C_p(z_1, z_2, \ldots, z_n) = \text{det}
\begin{bmatrix}
I_{m_1} - z_1 A_{11} & -z_1 A_{12} & \cdots & -z_1 A_{1n} \\
-z_2 A_{21} & I_{m_2} - z_2 A_{22} & \cdots & -z_2 A_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
-z_n A_{n1} & -z_n A_{n2} & \cdots & I_{m_n} - z_n A_{nn}
\end{bmatrix}
\]

The \( n \)-D stability condition requires that the characteristic polynomial has no zeros inside the closed \( n \)-disk \( \mathbb{C}^n \), i.e.,

\[
C_p(z_1, z_2, \ldots, z_n) \neq 0 \text{ in } \mathbb{C}^n
\]  

(7)
where $\bar{U}^n = \{(\bar{z}_1, \ldots, \bar{z}_n) | |\bar{z}_i| \leq 1, \ldots, |\bar{z}_n| \leq 1\}.$

To proceed further in a compact way, we define some matrices first. Let $A_1(i)$ be a submatrix obtained by deleting the $i$th block row and $j$th block column of $A$. $A_2(i)$ be the $j$th block column of $A$ with deleting $A_{ii}$, $A_3(i)$ be the $i$th block row of $A$ with deleting $A_{ii}$, for $i = 1, 2, \ldots, n$. In the case of $i = 2, \ldots, n - 1$, they can be expressed as follows:

$$A_1(i) = \begin{bmatrix}
A_{11} & \cdots & A_{i-1, i-1} & A_{i, i+1} & \cdots & A_{i, n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
A_{i-1, i} & \cdots & A_{i-1, i-1} & A_{i-1, i+1} & \cdots & A_{i-1, n} \\
A_{i+1, i} & \cdots & A_{i+1, i-1} & A_{i+1, i+1} & \cdots & A_{i+1, n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
A_{n1} & \cdots & A_{n, i-1} & A_{n, i+1} & \cdots & A_{nn}
\end{bmatrix}$$

(8)

$$A_2(i) = \begin{bmatrix}
A_{1i} \\
A_{2i-1, i} \\
\vdots \\
A_{ni}
\end{bmatrix}$$

(9)

$$A_3(i) = \begin{bmatrix}
A_{i1} & \cdots & A_{i, i-1} & A_{i, i+1} & \cdots & A_{in}
\end{bmatrix}.$$  

(10)

When $i = 1$ or $i = n$, the first or last block row and block column are deleted and it is easy to express the resulting matrices $A_1(i)$, $A_2(i)$ and $A_3(i)$ in a similar way as in (8)–(10). Now let

$$A(i) = \begin{bmatrix}
A_1(i) & A_2(i) \\
A_3(i) & A_{ii}
\end{bmatrix}$$

(11)

and

$$\Lambda_i = T_1 \oplus \cdots \oplus T_{i-1} \oplus T_{i+1} \oplus \cdots \oplus T_n, \quad 1 \leq i \leq n$$

(12)

where $T_j \in R^{m_j \times m_j}$, $j = 1, \ldots, n$. It is straightforward to generate the above matrices for any $n$ and any $i = 1, \ldots, n$.

Theorem 1: Consider an $n$-D digital system described by (4). For some nonsingular $\Lambda_i$, see (12), $S(z_i^{-1})$ given by

$$S(z_i^{-1}) = \Lambda_i [A_1(i) + A_2(i) (z_i^{-1} I - A_{ii}^{-1}) A_3(i)] \Lambda_i^{-1}$$

(13)

is strictly bounded real and $\rho(A_{ii}) < 1$ if and only if there exist some positive definite matrices $P_{ij} \in R^{m_i \times m_j}$, $j = 1, \ldots, n$, and $Q$ such that $P = P_{11} \oplus \cdots \oplus P_{nn}$ and $Q$ satisfy the $n$-D Lyapunov equation

$$P - A^T P A = Q.$$  

(14)

Remark 2: This theorem is an extended version of Theorem 2 in [7] (and Theorem 1 in [6] for 2-D case). In [7] (and [6]), the reachability and observability of $(A_{mn}, A_3(n))$ and $(A_{nn}, A^{T}_m(n))$ respectively is required. This is a stronger condition than $\rho(A_{ii}) < 1$ and thus the new condition can be used for a larger class of $n$-D systems.

Proof: The proof of this theorem is an extension of those in [6] and [7], details are given below:

Necessity: If $\Lambda_i$ is nonsingular, $S(z_i^{-1})$ is strictly bounded real and $\rho(A_{ii}) < 1$, then from Lemma 1 follows that there exists a positive definite matrix $P_{ii}$ such that the matrix in (15) shown at the bottom of the page is positive definite. Premultiplication of (15) with $(\Lambda_i^T \oplus I_{m_i})$ and postmultiplication with $(\Lambda_i \oplus I_{m_i})$ yields

$$Q_i = P_{ii} - A^T_{ii} P_{ii} A_{ii}$$

(15)

$$Q_i = \begin{bmatrix}
A_{1i} & A_{2i} \\
A_{3i} & A_{ii}
\end{bmatrix}$$

(16)

Let $U$ be a matrix as

$$U = \begin{bmatrix}
I_{m_1} \oplus \cdots \oplus I_{m_{i-1}} & 0 & 0 \\
0 & I_{m_i} & 0 \\
0 & 0 & I_{m_{i+1}} \oplus \cdots \oplus I_{m_n}
\end{bmatrix}.$$  

(17)

Noting that $U^{-1} = U^T$ and (8)–(12), (16) gives after premultiplication with $U$ and postmultiplication with $U^T$:

$$Q = P - A^T P A$$

(18)

which is the $n$-D Lyapunov equation

$$Q = U (A_i \oplus I_m)^T Q_i (A_i \oplus I_m) U^T$$

(19)

being positive definite.

Sufficiency: Suppose (14) is satisfied. Premultiplying (14) by $U^T$ and postmultiplying by $U$ gives

$$U^T Q U = \begin{bmatrix}
P_{ii} & 0 \\
0 & P_{ii}
\end{bmatrix} - \begin{bmatrix}
A_{1i} & A_{2i} \\
A_{3i} & A_{ii}
\end{bmatrix}^T
\begin{bmatrix}
P_{ii} & 0 \\
0 & P_{ii}
\end{bmatrix} \begin{bmatrix}
A_{1i} & A_{2i} \\
A_{3i} & A_{ii}
\end{bmatrix}$$

(20)

with $P_i = P_{i1} \oplus \cdots \oplus P_{i-1, i-1} \oplus P_{i, i+1} \oplus \cdots \oplus P_{im}$. Let $\Lambda_i = P_i^{1/2}$, $F = A_{ii}$, $G = A_3(i) \Lambda_i^{-1}$, $H^T = \Lambda_i A_2(i)$, and $J = \Lambda_i A_3(i)$, then premultiplying (21) by $(\Lambda_i^T \oplus I_{m_i})$ and postmultiplying by $(\Lambda_i^T \oplus I_{m_i})$, we can obtain that $S(z_i^{-1})$ is strictly bounded real and $\rho(A_{ii}) < 1$ from Lemma 1. □
Theorem 2: Consider an \( n \)-D digital system described by (4). Then, \( S(z_j^{-1}) \) given by

\[
S(z_j^{-1}) = \Lambda_j[A_2(j) + A_2(j)(z_j^{-1}I - A_3(j))^{-1}A_3(j)]
\]

is strictly bounded real for some nonsingular \( \Lambda_j \), see (12), and \( \rho(A_{ij}) < 1 \) if and only if there exist some nonsingular \( \Lambda_{zi} \), \( i \neq j \), see (12), so that \( S(z_i^{-1}) \) given by (13) is strictly bounded real and \( \rho(A_{zi}) < 1 \).

Proof: Both the sufficiency and necessity can be easily established by Theorem 1.

Theorems 1 and 2 present the rigorous necessary and sufficient conditions for the \( n \)-D Lyapunov equation and they do not involve reachability and observability conditions. It was shown in [6] that this means that the characteristic polynomial of the \( n \)-D system contains 1-D factors. In the rest of this section two theorems will be presented which will show that if an \( n \)-D system does not satisfy the reachability and/or observability conditions in some \( z_i^{-1} \) dimensions, then the original \( n \)-D system can be decomposed into \((m+1)\) subsystems, one is an irreducible \( k \)-D \((0 \leq k \leq n)\) subsystem, others are \( m \) \((1 \leq m \leq n)\) 1-D subsystems. The original \( n \)-D system has a positive definite solution to the \( n \)-D Lyapunov equation if and only if all the 1-D subsystems are stable and the irreducible \( k \)-D subsystem has a positive definite solution to the \( k \)-D Lyapunov equation. These results answer the questions left open in [6] and [7] and also clarify some inaccurate statements made there. Further, based on Theorems 3 and 4 a simple algorithm is developed to obtain the positive definite solution to the \( n \)-D Lyapunov equation based on the positive definite solutions of the \( m \) 1-D and the \( k \)-D systems. This simplifies the complexity of solving the \( n \)-D Lyapunov equation when the \( n \)-D system does not satisfy the reachability and/or observability conditions.

Theorem 3: If \( (A_{ii}, A_3(i)) \) is not reachable and/or \( (A_{ii}, A_3^T(i)) \) is not observable, there always exists a nonsingular matrix

\[
R = T_1 \oplus T_2 \oplus \cdots \oplus T_n
\]

to decompose the original \( n \)-D system into \((m+1)\) \((1 \leq m \leq n)\) subsystems. One of them is a \( k \)-D \((0 \leq k \leq n)\) subsystem which satisfies the reachability and observability conditions in those \( k \) dimensions, others are \( m \) 1-D subsystems in the \( m \) dimensions.

To prove the above theorem, the following proposition will be used.

Proposition 1: If \( (A_{ii}, A_3(i)) \) is not reachable and/or \( (A_{ii}, A_3^T(i)) \) is not observable, there always exists a nonsingular matrix

\[
R_i = I_{m_{i1}} \oplus \cdots \oplus I_{m_{i(m-1)}} \oplus I_{m_{i+1}} \oplus \cdots \oplus I_{m_{in}}
\]

to decompose the original \( n \)-D system into two subsystems. One is a 1-D subsystem and the other is an \( n \)-D subsystem or \((n-1)\)-D subsystem.

Proof: Set \( A(i) = U^T A U \), then \( A(i) \) has the form as (11). Based on the results of [11, pp. 130–132], we have the following.

i) If \( (A_{ii}, A_3(i)) \) is not reachable, then there always exists a nonsingular matrix

\[
R(i) = U^T R_k U
\]

\[
= I_{m_{i1}} \oplus \cdots \oplus I_{m_{i(m-1)}} \oplus I_{m_{i+1}} \oplus \cdots \oplus I_{m_{in}} \oplus T_i
\]

such that

\[
\hat{A}(i) = R^{-1}(i) A(i) R(i)
\]

\[
= \begin{bmatrix}
A_1(i) & A_{21}(i) & A_{22}(i) \\
A_{31}(i) & A_{3i1} & A_{3i2}
\end{bmatrix}
\]

(26)

where \((A_{ii}, A_3(i))\) is reachable, or \( A_{ii1} \) is null.

ii) \( (A_{ii}, A_3^T(i)) \) is not observable, then there always exists a nonsingular matrix \( R(i) \) given by (25) such that

\[
\hat{A}(i) = R^{-1}(i) A(i) R(i)
\]

\[
= \begin{bmatrix}
A_1(i) & A_{21}(i) & 0 \\
A_{31}(i) & A_{3i1} & A_{3i2} & A_{3i3}
\end{bmatrix}
\]

(27)

where \((A_{ii}, A_3^T(i))\) is observable, or \( A_{ii1} \) is null.

iii) If \( (A_{ii}, A_3(i)) \) is not reachable and \( (A_{ii}, A_3^T(i)) \) is not observable, then there always exists a nonsingular matrix \( R(i) \) given by (25) such that

\[
\hat{A}(i) = R^{-1}(i) A(i) R(i)
\]

\[
= \begin{bmatrix}
A_1(i) & A_{21}(i) & 0 & x & 0 \\
A_{31}(i) & A_{3i1} & A_{3i2} & A_{3i3} & 0 \\
x & x & x & x & x
\end{bmatrix}
\]

(28)

where \((A_{ii}, A_3(i))\) is reachable and \((A_{ii}, A_3^T(i))\) is observable, or \( A_{ii1} \) is null.

Transfer \( \hat{A}(i) \) back to \( \hat{A} = U \hat{A}(i) U^T \), we obtain the desired result immediately.

Theorem 3 can now be proven based on Proposition 1 as follows.

Step 1: Apply \( R(i) \) given by (25) to decompose the original \( n \)-D system into two subsystems, one is a reduced 1-D subsystem, the other is a reduced \( n \)-D or \((n-1)\)-D subsystem which system matrix is

\[
A_r = \begin{bmatrix}
A_1(i) & A_{21}(i) \\
A_{31}(i) & A_{3i1}
\end{bmatrix}
\]

(29)

Step 2: For the reduced \( n \)-D or \((n-1)\)-D subsystem \( A_r \), if the reachability and observability conditions are not satisfied in one (or more) dimension, then we can decompose this subsystem into two subsystems, one is a 1-D subsystem, the other is a \( k \)-D \((k = n, n-1, or n-2)\) subsystem.

Step 3: Repeat Steps 1 and 2 until the reduced \( k \)-D subsystem can not be decomposed any more (say irreducible \( k \)-D subsystem).

The proof of Theorem 3 is then completed.
Theorem 4: If \((A_{ii}, A_3(i))\) is not reachable and/or \((A_{ii}, A_3^T(i))\) is not observable, then the positive definite solutions to the Lyapunov equations for the reduced \(k\)-D \((0 \leq k \leq n)\) subsystem and the \(m\) \((1 \leq m \leq n)\) reduced 1-D subsystems exist, if and only if some positive definite solutions to the \(n\)-D Lyapunov equation for the original \(n\)-D system exist.

To prove the above theorem, the following proposition will be used.

Proposition 2: If \((A_{ii}, A_3(i))\) is not reachable and/or \((A_{ii}, A_3^T(i))\) is not observable, then positive definite solutions to the Lyapunov equations for both the reduced 1-D and the reduced \(n\)-D subsystems exist, if and only if some positive definite solutions to the \(n\)-D Lyapunov equation for the original \(n\)-D system exist.

Proof: Set again \(A(i) = U^T A U\) given by (11) and (17).

Consider the following cases:

i) If \((A_{ii}, A_3(i))\) is not reachable, \((A_{ii}, A_3^T(i))\) is observable.

Sufficiency: Suppose (14) is satisfied. We have

\[
P(i) - A^T(i) P(i) A(i) = Q(i)
\]

which is the same as (21) with \(P(i) = P_i \oplus P_{ii}\) and \(Q(i) = U^T QU\).

As we know, there exists a nonsingular matrix \(R(i)\) given by (25) to transfer \(A(i)\) to \(\hat{A}(i)\) given by (26) and

\[
\hat{P}(i) = R(i)^T P(i) R(i) = \begin{bmatrix} P_i & 0 & 0 & 0 \\ 0 & P_{i1} & P_{i2} & 0 \\ \end{bmatrix}
\]

with \(P_{i1} \in R^{p \times p}\) and \(P_{i2} \in R^{(m-p) \times (m-p)}\).

Premultiplying and postmultiplying (30) by \(R(i)^T\) and \(R(i)\), respectively, yields

\[
\hat{P}(i) - \hat{A}^T(i) \hat{P}(i) \hat{A}(i) = R(i)^T Q(i) R(i).
\]

The top left \((\sum_{i=1}^{n-1} m_i + p) \times \sum_{i=1}^{n-1} m_i + p\) principal submatrix of \(\hat{P}(i) - \hat{A}^T(i) \hat{P}(i) \hat{A}(i)\) is given to be

\[
\hat{Q}_{1p} = \begin{bmatrix} P_i & 0 \\ 0 & P_{i1} & P_{i2} \\ \end{bmatrix} - \begin{bmatrix} A_i & A_{21}(i) \\ A_{31}(i) & A_{ii} \\ \end{bmatrix}^T 
\times \begin{bmatrix} P_i & 0 \\ 0 & P_{i1} & P_{i2} \\ \end{bmatrix} - \begin{bmatrix} A_i & A_{21}(i) \\ A_{31}(i) & A_{ii} \\ \end{bmatrix}.
\]

According to "any principle submatrix of a positive definite matrix is positive definite" [16, p. 397] together with (25), (26), and (32), we can conclude that there exist some positive definite matrices \(\hat{P}_{1p} = P_i \oplus P_{i1}\) and \(\hat{Q}_{1p}\) to satisfy the \(n\)-D Lyapunov equation for the reduced \(n\)-D system which satisfies the reachability and observability conditions.

Furthermore, from (14) and (26), we have \(\rho(A) < 1\) and \(\rho(A_{ii}) < 1\), therefore, there always exists a positive definite matrix \(\hat{P}_{1p}\) such that

\[
\hat{Q}_{1p} = \hat{P}_{1p} - A_i \hat{P}_{1p} A_i
\]

is positive definite [16, p. 410].

Necessity: Suppose (33) and (34) are satisfied, this means that the positive definite solutions to the Lyapunov equations for the both the reduced 1-D and the reduced \(n\)-D subsystems exist. Let \(\hat{P}(i) = (\hat{P}_{1p} \oplus \sigma_i^2 \hat{P}_{i2}), \hat{P}(i) = R(i)^T P(i) R(i)\) with \(\sigma_i \neq 0\) and \(R(i)\) given by (25). From (26) and (30), we have

\[
R(i)^T Q(i) R(i) = \begin{bmatrix} \hat{Q}_{1p} & -A_{i2}^T \hat{P}_{1p} A_{i2} \\ -A_{i2} \hat{P}_{1p} A_{i2} & \sigma_i^2 \hat{Q}_{2p} - A_{i2}^T \hat{P}_{i2} A_{i2} \end{bmatrix}
\]

with

\[
\hat{A}_{i2} = \begin{bmatrix} A_i & A_{21}(i) \\ A_{31}(i) & A_{ii} \end{bmatrix}, \quad \hat{A}_{i2} = \begin{bmatrix} A_{i2}(i) \\ A_{i2} \end{bmatrix}.
\]

If \(\sigma_i^2\) is large enough, \(\hat{Q}_r\) given by (36) is positive definite.

\[
\hat{Q}_r = \sigma_i^2 \hat{Q}_{2p} - A_{i2}^T \hat{P}_{i2} A_{i2} - A_{i2} \hat{P}_{i2} A_{i2} \hat{Q}_{i2}^{-1} A_{i2}^T \hat{P}_{i2} A_{i2}.
\]

Based on \(\hat{Q}_{1p}\) and \(\hat{Q}_r\) being positive definite and [16, Theorem 7.7.6], we conclude that \(R(i)^T Q(i) R(i)\) given by (35) is positive definite, thus \(Q\) given by (14) is positive definite if \(\sigma_i^2\) is large enough.

Similarly, we can prove this theorem: ii) if \((A_{ii}, A_3(i))\) is reachable and \((A_{ii}, A_3^T(i))\) is not observable, and iii) if \((A_{ii}, A_3(i))\) is not reachable and \((A_{ii}, A_3^T(i))\) is not observable.

At this stage, it is not hard to prove Theorem 4 based on Theorem 3 and Proposition 2. Details are omitted here for brevity.

For completeness the lemma proposed in [7] for comparing the stability condition (7) with the condition for the existence of positive definite solutions to the \(n\)-D Lyapunov equation, can be extended to:

Lemma 2: Consider an \(n\)-D digital system described by (4). If \(S(z_3^{-1})\), given by (13), is strictly bounded real and \(\rho(A_{ii}) < 1\), then the characteristic polynomial of system (4) has no zeros in the unit \(n\)-disk.

Proof: It is easy to prove this lemma based on Lemma 1 of [7], Propositions 1 and 2 in this paper.

Combining Theorem 1 and Lemma 2, we obtain the following result.

Corollary 1: Consider an \(n\)-D digital system described by (4), if there exist some positive definite solutions to the \(n\)-D Lyapunov equation (14), then the characteristic polynomial of system (4) has no zeros in the unit \(n\)-disk.

IV. ILLUSTRATIVE EXAMPLE

To demonstrate the applicability of the present results and compare them with previous results, we now consider a 4-D \((3, 2, 2, 3)\)th-order digital systems with \(A\) given by (37) at the bottom of the next page.

It is easy to check that there is not any state in \(z_3^{-1}\) dimension, which satisfies both the reachability and observability conditions and thus, we can treat the 3-D subsystem \(A_1(3)\) separately. For the subsystem \(A_1(3)\), the third state is unreachable in \(z_3^{-1}\) dimension, the second state in \(z_3^{-1}\) dimension is unreachable and the first state in \(z_3^{-1}\) dimension is unobservable. Therefore, we can get an irreducible 3-D (2,
1. 2th-order subsystem from $A_1(3)$

$$A_r = \begin{bmatrix}
0.4 & 0.3 & 0 & 1 & 0 \\
0.3 & 0 & 0.1 & 0 & 0 \\
-0.1 & 0 & 0.3 & 0.1 & 0.1 \\
0.1 & 0.1 & 0 & 0.1 & 0 \\
-0.04 & -0.2 & 0.1 & 0.2 & 0.1
\end{bmatrix}. \tag{38}$$

Note that all the unreachable and/or unobservable states of $A$ are stable, therefore, we only need to test the stability of the 3-D (2, 1, 2)th-order subsystem $A_r$.

Using the method of [10], we obtain a positive definite matrix

$$P_r = \begin{bmatrix}
3 & 1 & 0 & 0 & 0 \\
1.5 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 6 & 1 \\
0 & 0 & 0 & 1 & 7
\end{bmatrix}. \tag{39}$$

such that

$$Q_r = P_r - A_r^TP_rA_r = \begin{bmatrix}
1.035 & -0.01 & 0.215 & -0.97 & 0.08 \\
-0.01 & 0.93 & 0.1 & -0.68 & -0.27 \\
0.215 & 0.1 & 0.825 & -0.28 & -0.2 \\
-0.97 & -0.68 & -0.28 & 2.61 & 0.04 \\
0.08 & -0.27 & -0.2 & 0.04 & 0.72
\end{bmatrix}. \tag{40}$$

is positive definite. Hence the 4-D system given by (37) is stable.

Moreover, based on $P_r$ given by (39), and adjusting $\sigma_i$ (see the proofs of Proposition 2 and Lemma 1), we obtain a positive definite matrix

$$P = \begin{bmatrix}
3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 15 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7
\end{bmatrix}. \tag{41}$$

such that $Q = P - A^TPA$ is positive definite.

V. CONCLUSION

In this paper, the discrete-time strictly bounded-real lemma for nonminimal realizations of 1-D digital systems is presented. Based on this lemma, new conditions for the existence of positive definite solutions to the $\eta$-D Lyapunov equation have been presented and the relationship between stability and these new conditions is established. These new conditions do not involve reachability and observability requirements and thus can be applied to $\eta$-D system which have characteristic polynomials with 1-D factor polynomials. Theorem 2 shows that if there are no nonsingular matrices $\Delta_k$, given by (12), so that $S(z^{-1})$, given by (13), is strictly bounded real, then it is impossible for $S(z^{-1})$, given by (22), $i \neq j$, to be strictly bounded real and thus, no positive definite solutions to the $\eta$-D Lyapunov equation exist. Further, in Theorems 3 and 4 it is shown that the positive definite solutions to the $\eta$-D Lyapunov equation can be obtained from the positive definite solutions of the Lyapunov equations for an irreducible $k$-D $(0 \leq k \leq \eta)$ subsystem and $m (1 \leq m \leq \eta)$ 1-D subsystems. These results could significantly simplify the complexity of solving the $\eta$-D Lyapunov equation for such cases. The applications of them to get improved lower bounds for the stability margin of $\eta$-D digital systems, and to get tighter lower bounds for coefficient wordlengths for finite wordlength implementations of $\eta$-D systems with guaranteed stability were presented in [17]. Moreover, the application of the new results to present the discrete-time lossless bounded real lemma for $\eta$-dimensional digital systems is currently under preparation.

APPENDIX

PROOF OF LEMMA 1

**Proof:** If $\{F, G, H, J\}$ is a minimal realization of $S(z^{-1})$, then this Lemma is identical to Lemma 1 of [6]. Therefore, we consider that $\{F, G, H, J\}$ is a nonminimal realization of $S(z^{-1})$, which has three cases as follows:

i) $(F, G)$ is not reachable and $(F, H)$ is observable

In this case, there always exists a nonsingular matrix $T$ such that [11, p. 130]

$$\hat{F} = T^{-1}FT = \begin{bmatrix} F_1 & F_2 \\ 0 & F_3 \end{bmatrix} m - p \tag{A.1}$$

$$\hat{G} = T^{-1}G = \begin{bmatrix} G_1 \\ 0 \end{bmatrix} m - p \tag{A.2}$$

\[ A_r = \begin{bmatrix}
0.4 & 0.3 & 0 & 0.1 & 0 & 0.3 & 0 & 1 & 0 \\
0.3 & 0 & 0.2 & 0.1 & -0.2 & 0 & 0.2 & 0 & 0 \\
0 & 0.5 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0 \\
-0.1 & 0 & -0.2 & 0.3 & -0.1 & 0 & 0.3 & 0 & 0.1 \\
0 & 0 & 0 & 0.4 & 0 & 0.1 & 0 & 0 & 0 \\
0 & -0.2 & 0.1 & 0.1 & 0.3 & 0.2 & 0 & -0.1 & 0.1 \\
0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 \\
0.1 & 0.2 & 0 & 0.1 & -0.1 & 0 & 0 & 0.4 & 0.1 \\
0.1 & 0.1 & 0 & 0 & 0 & 0.1 & 0 & 0.1 & 1 \\
-0.4 & -0.2 & 0.1 & 0.1 & -0.2 & 0 & -0.1 & 0.2 & 0.1
\end{bmatrix}. \tag{37} \]
and

\[ \hat{H}^T = H^T T = [H_1^T \quad H_2^T] \quad (A.3) \]

with \((F_1, G_1)\) is reachable.

Based on (A.1)–(A.3) together with (2), \(S(z^{-1})\) can be given

\[ S(z^{-1}) = J + H_1^T (z^{-1} I - F_1)^{-1} G_1 \quad (A.4) \]

**Sufficiency:** There exists a positive definite matrix \(P\) such that \(Q_1\) given by (3) is positive definite. Premultiplying and postmultiplying (3) by \((I \oplus T^T)\) and \((I \oplus T)\), respectively, yields

\[ \hat{Q}_1 = \begin{bmatrix} I - J^T J - G^T P G & -(F^T P G + \hat{H} J)^T \\ -F^T P G & -F^T PP - HH^T \end{bmatrix} \quad (A.5) \]

where \( \hat{Q}_1 = (I \oplus T^T)Q_I(I \oplus T) \) and \( \hat{P} = T^T P T \).

Equation (A.5) gives after some algebraic manipulations

\[ \hat{Q}_1 = \begin{bmatrix} I - J^T J - G^T P_1 G_1 & -(F^T P_1 G_1 + H_1 J)^T \\ -F^T P_1 G_1 - H_1 J & P_1 - F^T P_1 F_1 - H_1 H_1^T \end{bmatrix} \quad (A.6) \]

where \( P_1 \) is the top left \( p \times p \) principal submatrix of \( \hat{P} \) and \( x \) denotes any matrix.

Using the fact that any principle submatrix of a positive definite matrix is positive definite [16, p. 397], then there exists a positive definite \( P_1 \) such that the matrix

\[ \hat{Q}_{11} = \begin{bmatrix} I - J^T J - G^T P_1 G_1 & -(F^T P_1 G_1 + H_1 J)^T \\ -F^T P_1 G_1 - H_1 J & P_1 - F^T P_1 F_1 - H_1 H_1^T \end{bmatrix} \quad (A.7) \]

is positive definite. Therefore, \( S(z^{-1}) \) is strictly bounded real [6].

Since \( P > 0 \) and \( P - F^T PF - HH^T > 0 \), we have \( \rho(F) < 1 \).

**Necessity:** \( S(z^{-1}) \) is strictly bounded real and \( \rho(F) < 1 \). There exist two positive definite matrices \( P_1 \) and \( P_3 \) such that the matrices

\[ \hat{Q}_{11} = \begin{bmatrix} I - J^T J - G^T P_1 G_1 & -(F^T P_1 G_1 + H_1 J)^T \\ -F^T P_1 G_1 - H_1 J & P_1 - F^T P_1 F_1 - H_1 H_1^T \end{bmatrix} \quad (A.8) \]

and

\[ \hat{Q}_{13} = P_3 - F_3^T P_3 F_3 \quad (A.9) \]

are positive definite.

Let \( P = T^{-T} (P_1 \oplus \sigma^2 P_3) T^{-1} (\sigma \neq 0) \) and

\[ Q = \begin{bmatrix} I - J^T J - G^T P G & -(F^T P G + H J)^T \\ -F^T P G - H J & P - F^T PF - HH^T \end{bmatrix} \quad (A.10) \]

then (A.10) gives after some manipulations together with (A.1)–(A.3) (see A.11) at the bottom of the page).

There is always a \( \sigma^2 \) large enough, so that \( Q_\ell \) given by (A.12) is positive definite

\[ Q_\ell = \sigma^2 \hat{Q}_{13} - F_3^T P_3 F_3 - H_3 H_3^T - \hat{Q}_{12} \hat{Q}_1^{-1} \hat{Q}_{12}^T \quad (A.12) \]

where

\[ \hat{Q}_{12} = [F_3^T P_3 G_1 + H_2 J \quad F_3^T P_3 F_1 + H_2 H_1^T] \quad (A.13) \]

Based on \( \hat{Q}_{11} \) and \( Q_\ell \) being positive definite and Theorem 7.7.6 of [16], we can conclude that \((I \oplus T^T)Q(I \oplus T)\) given by (A.11) is positive definite, hence \( \bar{Q} \) given by (A.10) is positive if \( \sigma^2 \) is large enough.

ii) \((F, G)\) is reachable and \((F, H)\) is not observable

In this case there always exist [11, p. 132] a nonsingular matrix \( T \) such that

\[ \hat{F} = T^{-1} FT = \begin{bmatrix} F_1 & 0 \\ F_2 & F_3 \end{bmatrix} \quad m - q \quad (A.14) \]

\[ \hat{G} = T^{-1} G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \quad (A.15) \]

and

\[ \hat{H} = T^T H = \begin{bmatrix} H_1 \\ 0 \end{bmatrix} m - q \quad (A.16) \]

with \((F_1, H_1)\) is observable.

The proof in this case is essentially as case i).

iii) \((F, G)\) is not reachable and \((F, H)\) is not observable

In this case there always exist [11, p. 132] a nonsingular matrix \( T \) such that

\[ \hat{F} = T^{-1} FT = \begin{bmatrix} F_1 & 0 & x \\ x & x & x \\ 0 & 0 & x \end{bmatrix} \quad (A.17) \]

\[ \hat{G} = T^{-1} G = \begin{bmatrix} G_1 \\ 0 \end{bmatrix} \quad (A.18) \]

with \((F_1, G_1)\) is reachable and \((F_1, H_1)\) is observable.

Based on the discussions of i) and ii), it is not difficult to prove this case.

At this stage, the desired result is completely proven. \( \Box \)
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