Criterion for light localization in random amplifying media

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Abstract

Dimensionless conductance for light propagating through a random medium with amplification tends to diverge with an increase of gain. This raises questions on the applicability of the localization criteria based on this quantity. To circumvent this problem, we study the properties of the ratio between the transmission (conductance) and the energy stored in the random medium. We argue that the generalized conductance $g_G = g\cdot(\mathcal{E}_0/\mathcal{E})$ – conductance normalized by the energy buildup (ratio between energy stored in the medium with gain $\mathcal{E}$ to that in the passive system $\mathcal{E}_0$) – may be a convenient quantity on which a localization criterion can be built.

Key words: Wave propagation in random media, Anderson localization, Amplification, Random lasers

PACS: 42.25.Dd, 42.25.Bs, 42.55.Zz

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Preprint submitted to Physica B *June 21, 2009*

1. Introduction

Anderson localization [1] (AL) of electrons can be understood as the effect due to repeated self-interference of de Broglie waves during their propagation in a random medium and the resulting cessation of diffusion[2, 3]. Particle conservation, enforced because the carriers have charge, lies in the foundation of the concept of AL.

Since it has been recognized that AL is a wave phenomenon[4, 5, 6], there has been a number of experimental studies of *light* localization [7, 8, 9, 10, 11, 12]. Understanding the effect of dissipation[5], ubiquitous in optical systems, turned out to be essential for proper physical description and interpretation of experimental results. It also prompted the search[10] for an alternative criterion of localization in absorbing media. Coherent amplification, which leads to an altogether new physical phenomenon of random lasing[13], demands further refinement of the concept of AL and its criteria in active random media.

For passive systems, dimensionless conductance, g, averaged over an ensemble of macroscopically equivalent, but microscopically different disorder realizations, is closely related to a number of criteria used to define the onset of localization [14, 15]. It has deep physical roots in the scaling theory of localization [15, 16], where g uniquely determines the scaling function describing mesoscopic transport through random medium.

Transmittance is [17, 18, 19] the electromagnetic counterpart of conductance. This analogy with mesoscopic electronic transport makes it tempting to adopt the localization criteria (LC) based on g in passive systems. However, the LC developed for passive systems may not be applicable for random media with gain/absorption. Indeed, in dissipative systems, $g \ll 1$ may not be indicative of the presence of localization [20, 10]. Likewise, $g \gg 1$ in an amplifying random medium may not necessarily preclude localization effects [21, 22].

In case of absorption, an alternative criterion, based on the magnitude of *fluctuation* of transmission normalized by its average, was put forward [10]. When gain is present, such an approach presents a fundamental problem. Indeed, there exists a non-zero probability of encountering a special realization within the ensemble where the given value of the gain parameter exceeds threshold for random lasing [23] (TRL). Without saturation effects, such realizations make the statistics ill-defined. Inclusion of the saturation introduces a dependence on system- and material-specific parameters which are not associated with wave-transport properties of the random medium. To avoid such dependence, and at the same time to regularize the statistical ensemble, in Ref. [23] we introduced conditional statistics by excluding the diverging contributions. We found that correlation linewidth $\delta\omega$ obtained in such ensemble, can be used to define Thouless parameter $\delta = \delta\omega/\Delta\omega$ in random medium with gain. Here $\Delta\omega$ is the average mode spacing which is equal to the reciprocal of the density of states in the system. It was shown [23] that $g = \delta$ relationship does not hold in non-passive systems. Nonetheless, the decrease of δ with an increase of gain strength correlated with enhancement of fluctuations of conductance normalized by its average [21]. These observations motivated us to explore analogies between the effects of amplification and localization in random media.

Although the conditional statistics approach turned out to be fruitful, it may be justified in weakly scattering media and when the gain parameter is far from the diffusive TRL. In strongly scattering media where sample-to-sample fluctuations are pronounced, a different approach is needed.

In this work, we investigate another way of regularizing statistics of transport through random media with amplification, with the goal of identifying a quantity on which the localization parameter can be based. When taken separately, both transmittance (conductance) and the energy inside the system $\mathcal{E} = \int_V W(\mathbf{r},\omega)dV$ exhibit the divergent behavior as TRL is approached. Here, $W(\mathbf{r},\omega) = n^2(\mathbf{r},\omega)\ E^2(\mathbf{r},\omega)$ is electromagnetic energy density expressed in terms of the refractive index and the local electric field. To counterbalance the divergent behavior of the transmittance, we propose to form a ratio, $g_G \propto T/\mathcal{E}$, between T and $\mathcal{E}$ from the same disorder realization. In Section 2 we study the properties of the ratio $T/\mathcal{E}$ in diffusive and localized regimes. In Section 3, based on $T/\mathcal{E}$, we introduce generalized conductance and discuss its properties.

2. $T/\mathcal{E}$ ratio in random medium

2.1. Diffusive regime in slab geometry

Lets consider the effect of gain on T and $\mathcal{E}$ in an optically thick ($\ell \ll L$) slab of 3D random medium described by the diffusive equation. The slab thickness is L, $D = c\ell/3$ is the diffusion coefficient, c – speed of light, ℓ – transport mean free path; $l_g = \tau_g c$ is (ballistic) gain length. The transmission and reflection coefficients can be directly obtained from the solution for an absorbing slab in Ref. [24] through formal substitution $l_a = -l_g$. Such treatment of gain in a scattering problem has become known as the "neg-

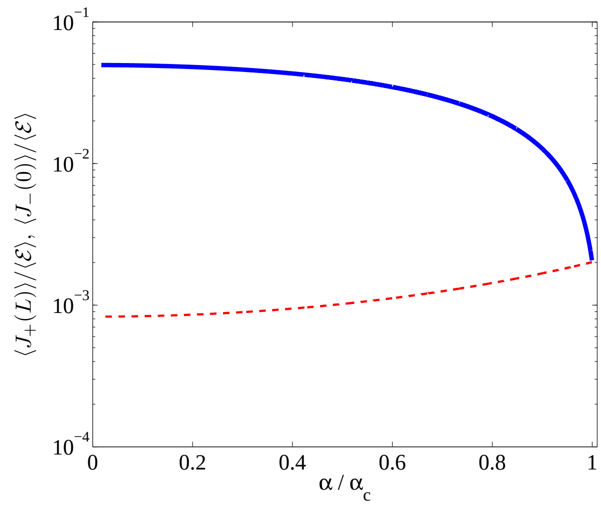

Figure 1: Transmission $\langle J_+(L)\rangle$ and reflection $\langle J_-(L)\rangle$ fluxes normalized by the value of total energy stored inside random medium (solid and dashed lines respectively) are plotted as functions of gain strength for a slab of random medium of thickness $L/\ell = 100$. The divergence in the vicinity of TRL is prevented due to the normalization – both curves approach the same limiting value.

ative absorption" model. It has been successfully used to describe turbid amplifying medium such as incoherent random lasers [25, 26, 27].

In Fig. 1 we plot the ratios of reflection (flux) to energy $\langle J_{-}(0)\rangle/\langle\mathcal{E}\rangle$ and transmission (flux) to energy $\langle J_{+}(L)\rangle/\langle\mathcal{E}\rangle$. For brevity we will refer to these quantities as $\langle R\rangle/\langle\mathcal{E}\rangle$ and $\langle T\rangle/\langle\mathcal{E}\rangle$. Within framework of the above model, we make the following observations: (a) Sufficiently close to TRL, the reflection and transmission fluxes diverge and become almost equal. This signifies the fact that the system approaches the regime when the gain alone can sustain its energy, without relying on the incident flux; (b) When normalized by the total energy in the slab, both $\langle R\rangle/\langle\mathcal{E}\rangle$ and $\langle T\rangle/\langle\mathcal{E}\rangle$ do not diverge when the TRL is approached. Instead, they converge to the finite value of $2\pi D\alpha_c$ (where $\alpha^{-1} = \sqrt{\ell l_g/3}$ and $\alpha_c \approx \pi/L$); (c) The change of the quantities $\langle R\rangle/\langle\mathcal{E}\rangle$ and $\langle T\rangle/\langle\mathcal{E}\rangle$ is related to modification of the intensity distribution inside the volume of random medium. When intensity $\langle I(z)\rangle$ assumes the limiting profile given by the lowest order diffusion mode, $\langle I(z)\rangle \simeq \sin(\pi z/L)$, the ratios $\langle R\rangle/\langle\mathcal{E}\rangle$ and $\langle T\rangle/\langle\mathcal{E}\rangle$ saturate.

As we are interested in the interplay between the effects of gain and light localization, we note that the diffusive description of this section has limitations: (i) the diffusion approximation fails when wave phenomena such as localization or coherent random lasing become important – proper treatment of electric field and its phase becomes necessary; (ii) an increase of gain or scattering are expected to lead to a buildup of fluctuations of transport coefficients [28, 23]. Thus, quantity, e.g., $\langle T\rangle/\langle\mathcal{E}\rangle$ will no longer adequately represent $T/\mathcal{E}$ and instead should be replaced with $\langle T/\mathcal{E}\rangle$, which accounts for correlation between T and $\mathcal{E}$ in the same sample; (iii) with further increase of gain toward TRL, the divergence of fluctuations of T may necessitate the consideration of higher moments of $T/\mathcal{E}$ or, perhaps, its entire distribution; (iv) at the onset of random lasing, nonlinear [29] and dynamical [30] processes become essential for proper description of the system properties and thus, CW quantity such as $\langle T/\mathcal{E}\rangle$ may no longer be suitable.

2.2. Localized regime in one-dimension

To investigate the effects (i-iii) from the previous section on $T/\mathcal{E}$, we consider a system in localized regime . For this purpose a one-dimensional (1D) model is already sufficient. Long enough 1D systems are necessarily in the localized regime and, therefore, fluctuation effects will be essential even at small values of gain.

We model the normal propagation of EM wave through a stack of dielectric slabs (a 1D system) using 2×2 transfer matrices[31, 32, 33, 34]. The randomness is introduced through the fluctuations of the refractive indexes of the slabs whereas the imaginary part of the index gives rise to linear gain. We use this numerical model to simulate the continuous-wave (CW) response of the random system within certain spectral range.

Motivated by our analysis in section 2.1, we would like to study the dependence of the ratio between transmission and stored energy in the above wave-model. Our analysis shows that T and $\mathcal{E}$ are not closely correlated in the localized regime. We find substantially more resonant peaks in the transmission coefficient with only about half of which have their counterparts in the energy. This disparity is an additional source of fluctuation in the ratio $T/\mathcal{E}$ [34]. The goal of this section is to understand this behavior.

The field distribution inside the sample gives a clue why the energy may differ from resonance to resonance. At the off-resonant frequencies we observe nearly exponential decay. Whereas at or in the vicinity of a tunneling resonance, two qualitatively distinct behaviors are observed. They are illustrated in Fig. 2.

In the first scenario, c.f. bold line in Fig. 2, the electric field grows exponentially (the characteristic length is equal to the localization length, ξ) towards the localization center x_0 and falls off after it. Such behavior is attributed[35] to the phenomenon of resonant tunneling via a localized state centered at x_0.

In the other case, c.f. thin line in Fig. 2, an additional negative exponential segment can be identified. Because this type of behavior leads to significantly lesser amount of energy stored inside the system, the resonances of this type do not show a pronounced spectral peak in $\mathcal{E}(\omega)$. Although the localized states with a single cusp in their spatial profiles were studied in Ref. [35], the second scenario exemplified by thin line in 2 was not described in the studies by Azbel and coworkers. We note that a multi-peaked spatial intensity distribution is expected in case of so-called necklace states[36, 37, 38] when two or more resonant states coexist at (almost) the same energy in the given disorder realization. This behavior has lower probability of occurrence in $\xi \ll L$ regime compared to the single states as in Fig. 2.

We find [34] that the difference in two types of behavior in Fig. 2 originates from the spatial location of the localized state: (i) at the frequencies where peaks in transmission and energy occur simultaneously, the center of localization is located in the first half of the sample $0 < x_0 < L/2$ (bold line in Fig. 2); (ii) at the frequencies where peak in transmission has significantly

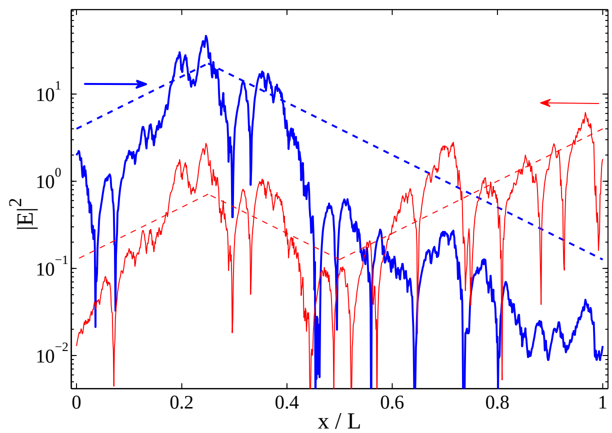

Figure 2: Two types of the on-resonance electric field distribution inside random medium with the center of localization in the first half (bold lines) and the second half (thin lines) of the sample. In the second case, the same mode is excited by shining the light onto the system from the right. Due to reciprocity, the value of transmission coefficient for both cases is exactly the same. However, the amount of energy stored inside the system is exponentially smaller in the second case. The latter resonance does not show noticeable peak in $\mathcal{E}(\omega)$. The dashed lines are the schematic envelope functions formed from segments with $\exp(\pm x/\xi)$ spatial dependences.

less pronounced (or non-distinguishable) peak in energy, the center of localization is located in the second half of the sample $L/2 < x_0 < L$ (thin line in Fig. 2); (iii) both profiles can be observed in the same sample at the same frequency by illuminating the system from left or right, see Fig. 2).

The non-monotonous behavior of the wavefunction can be understood intuitively as follows. In the regions away from a localization center the waves propagate via tunneling. There exist local solutions of Maxwell equation with an exponentially increasing and an exponentially decreasing envelopes [39]. Balance between these two components is determined from the boundary conditions. It appears [34] that in $L/2 < x_0 < L$ case, the exponentially increasing component has very small magnitude at the left boundary, but becomes dominant at the turning point $x_T = 2x_0 - L$. In contrast, when the localization center is located in the first half of the sample, the exponentially increasing component is dominant right from the left boundary of the sample.

Our analysis above shows that $T/\mathcal{E}$ parameter may not be unique in a given sample – in general, it depends on the direction of illumination. This observation seems to suggest that such ratio may be of a limited use as a localization parameter. However, this limitation can be overcome via simple normalization as we show in Section 3. Furthermore, in its present form, $T/\mathcal{E}$ possesses a unique property which we discuss next.

2.3. Evolution of $T/\mathcal{E}$ with the increase of gain

As we observed in Section 2.1, a change in $T/\mathcal{E}$ with gain is indicative of a modification of the intensity profile inside a diffusive slab. Similar conclusions can be made in the localized regime. Here we employ the model described in Section 2.2.

Our simulations demonstrate that a change in $T/\mathcal{E}$ indeed signifies the modification of the field distribution inside the random medium. Interestingly, we find that such modifications can be very dramatic in the localization regime. Fig. 3 illustrates this effect. At first glance, this seems to disagree with the conclusions in Refs. [40, 41, 42] where (in localized regime) little or no change in the field pattern was found with an increase of amplification. The apparent discrepancy can be explained if one compares the methods used to excite the system. In our work, we consider the transmission experiment setup, whereas in the previous works [40, 41, 42] the system was excited throughout its entire volume or relatively close to the center of localization.

(a) (b)

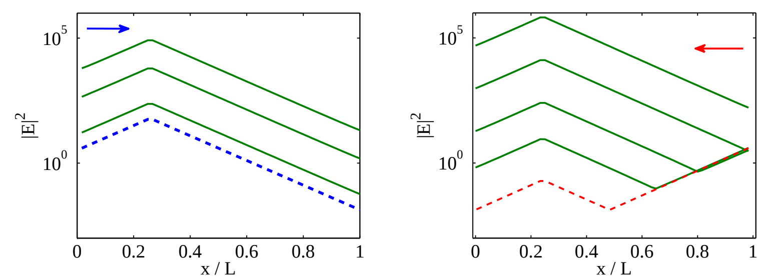

Figure 3: An illustration of the effect gain on the electric field distribution in 1D random medium. A periodic stack of alternating dielectric layers with a defect at $x_0 = L/4$ is considered. As demonstrated in Ref. [34], such model qualitatively describes the field profiles (such as those in Fig. 2) in the random media. Panels (a) and (b) show the field envelopes obtained when the system is illuminated (at resonant frequency) from the left and right respectively. Dashed lines correspond to the passive medium. The solid curves (from bottom up) are obtained for $l_{g,cr}/l_g$ equal to $0.5,\ 0.9,\ 0.99$ in (a) and $0.85,\ 0.95,\ 0.98,\ 0.99$ in (b). The field distribution in (b) shows a dramatic modification with an increase of gain.

Under such excitation conditions, the situation shown in Fig. 3a is always realized [34]. We also note that the mode distribution in Fig. 3b was observed to converge to that in Fig. 3a when the gain approached its critical value. when the field distribution is maintained by the gain with little reliance on the incident energy.

3. Discussion

Although the ratio $T/\mathcal{E}$ did not have the tendency to diverge when gain approached its TRL value, it exhibited additional fluctuation due to orientation of the system, Sections 2.2,2.3. It should be noted that due to reciprocity, transmission T (or more generally conductance) is independent of direction of illumination – it is the same for both field distributions depicted in Fig. 2.

To circumvent the above problem while retaining the desired non-singular behavior in the vicinity of TRL, we introduce the following parameter

$$g_G = T/\mathcal{E} \times \mathcal{E}_0. \quad (1)$$

Here $\mathcal{E}_0$ is the energy stored in the random medium with no gain. All the quantities entering Eq. (1) should be evaluated for each disorder realization prior to any statistical analysis. By construction, g_G reduces to the transmission in a 1D system without gain and becomes conventional (average) conductance upon statistical averaging in higher dimensional systems. Thus we will refer to g_G as generalized conductance.

By definition, the quantity $\mathcal{E}_0$ is equal to volume-integrated electromagnetic energy density in passive system. The latter coincides [43] with density of electromagnetic states, which is an important energy scale that enters such localization criteria as Thouless number δ. Therefore, the generalized conductance can be interpreted as the conductance in the random medium with gain scaled down by the energy buildup factor $\mathcal{E}/\mathcal{E}_0$. This normalization offsets the tendency of the conductance to diverge in a random medium with gain and avoids introducing artifacts such as the orientation dependence.

In future, we plan to investigate both theoretically and experimentally the statistical properties of the generalized conductance in Eq. (1) in random medium with gain. Experimentally, all components of g_G can be determined from near-field scanning measurements in two-dimensional random media – structurally disordered semiconductor films. This opens up a possibility to corroborate and extend the results of this study.

4. Acknowledgment

This work was supported by National Science Foundation Grant Nos. DMR-0704981 and DMR-0808937. AY is grateful to S. Skipetrov, A. Lagendijk and B. van Tiggelen for valuable comments.

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