Exercise 1. (2+1+1) Let $X$ be a non-empty set and $d : X \times X \rightarrow \mathbb{R}$ defined by

$$d(x,y) = \begin{cases} 
0 & x = y \\
1 & x \neq y 
\end{cases} \text{ for } x, y \in X.$$

(i) Show that $d$ is a metric on $X$.

(ii) Characterize the Cauchy sequences in $X$.

(iii) Show that $(X,d)$ is complete.

Exercise 2. (2+2+1) Consider the set $S$ defined by $S = \{1/n, n \in \mathbb{N}\}$.

(i) Is $S$ closed in $\mathbb{R}$ with respect to the standard metric? Is it open? Justify your answers.

(ii) Determine the interior, the closure and the boundary of the set $S$.

(iii) Is the set $\{n : n \in \mathbb{N}\}$ closed in $\mathbb{R}$?

Exercise 3. (2+2+1) Let $(X,d)$ be a metric space.

(i) Show that for $x_0 \in X$ and $r > 0$ the open ball $B(x_0, r) = \{y \in X : d(x_0, y) < r\}$ is indeed open.

(ii) Let $(x_n)$ and $(y_n)$ be two convergent sequences in $(X,d)$ with $x_n \rightarrow x$ and $y_n \rightarrow y$. Show that $d(x_n, y_n) \rightarrow d(x, y)$.

(iii) Let $A_\alpha \subset X$ be closed for any $\alpha \in I$, where $I$ is an arbitrary index set. Show that $\bigcap_{\alpha \in I} A_\alpha$ is closed.

Exercise 4. (1+1+1)

(i) Find a metric $d$ for the set $S = \{1/n, n \in \mathbb{N}\}$ so that $S$ equipped with this metric becomes a complete metric space.

(ii) Is the new space $(S,d)$ compact?

(iii) Show that the space $\mathbb{Q}$ of rational numbers with the standard metric is not complete.

The homework assignments are posted under the following address:

http://web.mst.edu/~akine/fall10/aa.html