Each of the first five problems is worth 20 points. Answer all questions in the space provided. In order to receive credit, your answer must show sufficient work to conclusions you obtain. Books and notes are not allowed. You might work on the last bonus question to have extra points.

1) For each question below, state the order and determine whether it is nonlinear, linear homogeneous or linear inhomogeneous. Provide reasons for your answers, i.e., show your work.

- \( u_{yy} - u_{xy} + u_{xxy} + uu_y = 0; \)

\[ \mathcal{L}u = u_{yy} - u_{xy} + u_{xxy} + uu_y \]

\[ \mathcal{L}(au) = au_{yy} - au_{xy} + au_{xxy} + au_{uy} \]

\[ = a(u_{yy} - u_{xy} + u_{xxy} + au_{uy}) \]

\[ = a \mathcal{L}u. \]

**Nonlinear, 3rd order**

- \( u_{xx} - u_y + yu + x = 0. \)

\[ \mathcal{L}u = u_{xx} - u_y + yu \Rightarrow \mathcal{L}u = -x. \]

\[ \mathcal{L}(u + v) = u_{xx} + v_{xx} - u_y - v_y + yu + yv \]

\[ = u_{xx} - u_y + yu + v_{xx} - v_y + yv \]

\[ = \mathcal{L}u + \mathcal{L}v. \]

\[ \mathcal{L}(au) = au_{xx} - auy + ya \]

\[ = a(u_{xx} - u_y + yu) \]

\[ = a \mathcal{L}u. \]

**Linear, 2nd order, inhom.**
2) Solve the partial differential equation

\[ tu_x + xu_t = 0 \text{ with } u(0,t) = e^{-t^2}. \]

\[
\frac{dt}{dx} = \frac{x}{t} \quad \Rightarrow \quad t \, dt = x \, dx \quad \Rightarrow \quad \frac{t^2}{2} = \frac{x^2}{2} + C
\]

\[ \Rightarrow \quad t = \sqrt{x^2 + C}, \quad 2C_1 = C \]

\[ \Rightarrow \quad t = \pm \sqrt{x^2 + C} \]

\[ \frac{du}{dx} u_x(t) = u_x \cdot 1 + u_t \cdot \frac{2x}{2\sqrt{x^2 + C}} = \frac{tu_x + xu_t}{t} = 0 \]

\[ \text{. . . u is constant on each characteristic curve} \]

\[ u(x,t) = u(x, \pm \sqrt{x^2 + C}) = u(0, \pm \sqrt{C}) = f(C), \]

where \( f \) is an arbitrary differentiable function of one variable.

\[ \therefore u(x,t) = f(t^2 - x^2) \]

Since \( u(0,t) = e^{-t^2} \), \( e^{-t^2} = f(t^2) \).

\[ \therefore u(x,t) = e^{-(t^2 - x^2)} = e^{x^2 - t^2} \]
3) Find the regions in the $xy$-plane where the equation

$$xu_{xx} - 4(x + y)u_{xy} + 18y u_{yy} + u_x - u = 0$$

is elliptic, hyperbolic or parabolic. Sketch them.

$$D = b^2 - 4ac = 16(x+y)^2 - 4 \times 18y$$

$$= 16(x^2 + 2xy + y^2) - 72xy$$

$$= 8(2x^2 - 5xy + 2y^2)$$

$$= 8(2x-y)(x-2y)$$

**Elliptic** if

$$0 < (2x-y)(x-2y)$$

**Hyperbolic** if

$$0 > (2x-y)(x-2y)$$

**Parabolic** if

$$(2x-y)(x-2y) = 0$$
4) Find the general solution of $u_{xx} - u_{xt} - 2u_{tt} = 0$ by “factoring” equation.

$$\left( \frac{\partial}{\partial x} - 2 \frac{\partial}{\partial t} \right) \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right) u = 0$$

Let $V_x - 2V_t = 0 \Rightarrow V(x,t) = f(-2x-t)$, $f$ is any differentiable arbitrary function of one variable.

$$\therefore \ u_x + u_t = f(-2x-t)$$

Let $\xi = -2x-t$

$\eta = x-t$

Then

$$u_x = -2u_\xi + u_\eta$$
$$u_t = -u_\xi - u_\eta$$

$$-2u_\xi + u_\eta - u_\xi - u_\eta = f(\xi)$$

$$u_\xi = -\frac{1}{3} f(\xi) \Rightarrow u = -\frac{1}{3} \int f(\xi) \, d\xi + G(\eta)$$

$$u = F(\xi) + G(\eta), \ F \ and \ G \ are \ arbitrary \ functions \ of \ one \ variable \ s + \ \xi = -\frac{1}{3} f.$$
5) Solve $u_{tt} = 16u_{xx}$ subject to the initial conditions $u(x, 0) = e^x$ and $u_t(x, 0) = 2 + x$.

\[
U(x, t) = \frac{1}{2} \left[ e^{x+4t} + e^{x-4t} \right] + \frac{1}{8} \int_{x-4t}^{x+4t} (2s) \, ds
\]

\[
= e^x \frac{e^{4t} + e^{-4t}}{2} + \frac{1}{8} \left( 2s + \frac{s^2}{2} \right)_{x-4t}^{x+4t}
\]

\[
= e^x \cosh 4t + \frac{1}{8} \left[ 2(x+4t) + \frac{1}{2} (x+4t)^2 - 2(x-4t) - \frac{1}{2} (x-4t)^2 \right]
\]

\[
= e^x \cosh 4t + \frac{1}{8} \left[ 8t + \frac{1}{2} (x^2 + 8xt + 16t^2) + 8t - \frac{1}{2} (x^2 - 8xt + 16t^2) \right]
\]

\[
= e^x \cosh 4t + \frac{1}{8} \left[ 16t + 8xt \right]
\]

\[
= e^x \cosh 4t + 2t + xt
\]
6) (10pts) Transform $u_{xx} + 2u_{xt} + u_{tt} = 0$ into standard form, and then obtain the general solution of the original partial differential equation.

Let $\xi = x \cos \alpha + t \sin \alpha$, $\eta = -x \sin \alpha + t \cos \alpha$, and $u(x,t) = W(\xi, \eta)$.

$u_x = \xi W_\xi + \eta W_\eta$, $u_\xi = \xi^2 W_\xi + 2\xi \eta W_\eta + \eta^2 W_\eta$,

$u_{xt} = \xi W_{\xi\eta} + (\xi^2 - \eta^2) W_\eta\eta - \xi \eta W_\eta$,

$u_t = \eta W_\xi + \xi W_\eta$, $u_{tt} = \eta^2 W_\xi + 2\eta \xi W_\eta + \xi^2 W_\eta$. 

$c^2 \xi W_{\xi\xi} - 2c \xi \eta W_{\xi\eta} + \eta^2 W_{\eta\eta} + 2c \xi \eta W_{\xi\eta} + 2(c^2 - s^2) \xi W_\eta - 2c \xi \eta W_\eta$ 

$+ \eta^2 W_\eta + 2c \xi \eta W_\eta + c^2 \eta W_\eta = 0$.

$(c^2 + 2cs + s^2) W_{\xi\xi} + (-2cs + 2c^2 - 2s^2 + 2c) W_{\xi\eta}$ 

$+ (s^2 - 2cs + c^2) W_{\eta\eta} = 0$.

We want

$c = s \Rightarrow \alpha = \frac{\pi}{4}$

$\Rightarrow 4c^2 W_{\xi\xi} = 0 \Rightarrow W_{\eta\eta} = 0 \Rightarrow W_\xi = f(\eta)$

$\Rightarrow W = f(\eta) + g(\eta)$

$\Rightarrow u(x,t) = (\xi + s\xi + ct) f(-\xi + ct) + g(-\xi + ct)$,

where $f$ and $g$ are arbitrary functions of one variable.

$u(x,t) = (\frac{\sqrt{2}}{2} x + \frac{\sqrt{2}}{2} t) f(-\frac{\sqrt{2}}{2} x + \frac{\sqrt{2}}{2} t) + g(-\frac{\sqrt{2}}{2} x + \frac{\sqrt{2}}{2} t)$. 