Sec. 2.3 Modeling with First Order Equations

4, p. 60: A tank with a capacity of 500 gallons originally contains 200 gallons of water with 100 pounds of salt in solution. Water containing 1 pound of salt per gallon is entering at a rate of 3 gallons per minute, and the mixture is allowed to flow out of the tank at a rate of 2 gallons per minute.

(a) Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow.

(b) Find the concentration (in pounds per gallon) of salt in the tank when it is on the point of overflowing. Compare this concentration with the theoretical limiting concentration if the tank had infinite capacity.

Solution: (a) We use

\[ \text{Net rate of change} = \text{Rate of inflow} - \text{Rate of outflow} \]

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If \( A(t) \) denotes the amount of salt (in pounds) in the tank at time \( t \) (in minutes) then

\[ \frac{dA}{dt} = \left( \frac{3 \text{ gallons}}{\text{minute}} \right) \left( \frac{1 \text{ pound}}{\text{gallon}} \right) - \left( \frac{2 \text{ gallons}}{\text{minute}} \right) \left( \frac{A(t) \text{ pounds}}{V(t) \text{ gallons}} \right) \]

where \( V(t) \) denotes the volume (in gallons) of water in the tank at time \( t \) (in minutes). Observe that the following data holds:

<table>
<thead>
<tr>
<th>( t )</th>
<th>( V(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>200</td>
</tr>
<tr>
<td>1</td>
<td>201</td>
</tr>
<tr>
<td>2</td>
<td>202</td>
</tr>
<tr>
<td>3</td>
<td>203</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( t )</td>
<td>200 + ( t )</td>
</tr>
</tbody>
</table>
Therefore $V(t) = 200 + t$ for the time interval $0 \leq t \leq 300$, after which the tank overflows. Consequently, the initial value problem that models the salt in the tank at time $t = 0$ to $t = 300$ is

$$\frac{dA}{dt} = 3 - \frac{2A}{200 + t}, \quad A(0) = 100.$$  

The differential equation is first order linear. Expressing it in standard form we have

$$\begin{align*}
(*) \quad \frac{dA}{dt} + \frac{2}{200 + t} A &= 3.
\end{align*}$$

An integrating factor is

$$\int p(t) \, dt = \int \frac{2}{200 + t} \, dt = 2 \ln(200+t) + C = \ln(200+t)^2,$$

where $C = 0$. Multiplying $(*)$ by the integrating factor gives

$$(**) \quad (200 + t)^2 \frac{dA}{dt} + 2 (200 + t) A = 3 (200 + t)^2.$$  

But the left member of this equation should be exact if we have done our work right. It should be

$$\frac{d}{dt} \left[ (200 + t)^2 A \right].$$

We use the product rule $(f \cdot g)' = f' \cdot g + f \cdot g'$ to check this,

$$\frac{d}{dt} \left[ (200 + t)^2 A \right] = (200 + t)^2 \frac{dA}{dt} + 2 (200 + t) A \checkmark$$

Therefore $(**)$ is equivalent to

$$(***) \quad \frac{d}{dt} \left[ (200 + t)^2 A \right] = 3 (200 + t)^2.$$
Integrating (**k**) produces
\[ \int \frac{d}{dt} \left[ (200+t)^2 A \right] \, dt = \int 3(200+t)^2 \, dt \]

\[ (200+t)^2 A = (200+t)^3 + C \]

\[ \therefore \quad A(t) = 200 + t + \frac{C}{(200+t)^2} \]

We apply the initial condition:

\[ 100 = A(0) = 200 + \frac{C}{(200)^2} \]

so \( C = -4,000,000 \). Consequently

\[ A(t) = 200 + t - \frac{4,000,000}{(200+t)^2} \]

over the time interval \( 0 \leq t \leq 300 \).

**b**. The concentration of salt in the tank at the instant \( t = 300 \) minutes when the tank begins to overflow is

\[ C = \frac{A(300)}{V(300)} = \frac{500 + \frac{-4,000,000}{(500)^2}}{500} = 1 - \frac{4 \times 10^6}{125 \times 10^6} \]

so \( C = -0.968 \) (or \( \frac{121}{125} \)) pounds per gallon.

Note that if the tank had an infinite capacity then the limiting value of the concentration is

\[ \lim_{t \to \infty} C(t) = \lim_{t \to \infty} \frac{A(t)}{V(t)} = \lim_{t \to \infty} \frac{200 + t - \frac{4,000,000}{(200+t)^2}}{200 + t} = 1 \]

pound per gal.
Sec. 2.5 Autonomous Equations and Population Dynamics

79, p. 89: Consider \( \frac{dy}{dt} = y^2(y^2 - 1) \), \(-\infty < y_0 < \infty\). Sketch the graph of \( f(y) = y^2(y^2 - 1) \), determine the critical (equilibrium) points, classify each one as asymptotically stable, unstable, or semistable. Draw the phase line and sketch several graphs of solutions in the \( ty \)-plane.

\[
\begin{align*}
&\text{The equilibrium points satisfy} \\
&0 = f(y) = y^2(y^2 - 1) = y^2(y+1)(y-1). \\
&\text{Therefore,} \quad y = 0, y = -1, \text{ and } y = 1 \quad \text{are the equilibrium points.}
\end{align*}
\]

<table>
<thead>
<tr>
<th>interval</th>
<th>sign of ( y' (= y^2(y^2 - 1)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\infty &lt; y &lt; -1)</td>
<td>+</td>
</tr>
<tr>
<td>(-1 &lt; y &lt; 0)</td>
<td>-</td>
</tr>
<tr>
<td>(0 &lt; y &lt; 1)</td>
<td>-</td>
</tr>
<tr>
<td>(1 &lt; y &lt; \infty)</td>
<td>+</td>
</tr>
</tbody>
</table>

Phase Line:
- \( y = 1 \) (unstable)
- \( y = 0 \) (semistable)
- \( y = -1 \) (asymptotically stable)
Some typical graphs of solutions $y = y(t)$ of the DE

$$y' = y^2(y^2 - 1).$$