Fixed Income Models

Exercise Sheet 2
(Due: Tuesday 11/16/2010)

6. (Ito’s formula)
   Let \( W \) be a Brownian motion. For arbitrary \( n \in \mathbb{N} \) find a formula for \( \int_0^t W^n(s)\,dW(s) \).

7. (Ito’s formula, Integration by parts)
   Let \( W \) be a Brownian motion. Prove the following “integration by parts” formula for \( n \in \mathbb{N} \):
   \[
   \int_0^t s^n\,dW(s) = s^nW(t) - \int_0^t ns^{n-1}W(s)\,ds.
   \]

8. (Brownian motion)
   Show that if \( W \) is a Brownian motion, then
   \[
   \mathbb{E}(W^2(t)) = t.
   \]

9. (Brownian motion, Covariance)
   Show that for a Brownian motion \( W \), the covariance between two time points \( s \) and \( t \) can be computed as
   \[
   \text{Cov}(W(s), W(t)) = \min\{s, t\}.
   \]

10. (Brownian motion)
    Let \( Y \) be a standard normally distributed random variable and define \( X(t) = \sqrt{t}Y \). Is the process \( X \) a Brownian motion?

11. (Log-normal distribution)
    Let \( Y \) be a normally distributed random variable with \( \mathbb{E}(Y) = \mu \) and \( \mathbb{V}(Y) = \sigma^2 \). Show that
    \[
    \mathbb{E}(e^Y) = e^{\mu + \frac{\sigma^2}{2}}.
    \]

12. (Generalized geometric Brownian motion)
    Compute
    \[
    \mathbb{E}(X(t)|\mathcal{F}(s))
    \]
    and
    \[
    \mathbb{V}(X(t)|\mathcal{F}(s)),
    \]
    where \( X \) is the solution of the SDE
    \[
    dX(t) = \rho(t)X(t)\,dt + \sigma(t)X(t)\,dW(t).
    \]

13. (Stochastic Leibniz rule)
    Let \( \mu \in \mathbb{R} \) and \( \sigma > 0 \). Consider the two processes \( Y \) and \( Z \) with
    \[
    dY(t) = Y(t)\left(\mu\,dt + \sigma\,dW(t)\right)
    \]
    and
    \[
    Z(t) = \exp\left(-\frac{1}{2}\sigma^2-t-\mu\sigma W(t)\right).
    \]
    Determine the SDE that is satisfied by \( YZ \).

14. (Option price)
    Let \( K > 0 \). Assume \( Y \) is a lognormally distributed random variable with \( \mathbb{E}(\ln(Y)) = M \) and \( \mathbb{V}(\ln(Y)) = V^2 \). Show that
    \[
    \mathbb{E}\left((Y - K)^+\right) = e^{M+\frac{V^2}{2}} \Phi\left(M - V\sqrt{\frac{M-K}{V}}\right) - K \Phi\left(M - \ln(K)\sqrt{\frac{M-K}{V}}\right),
    \]
    where \( \Phi \) is the cdf of the standard normal distribution.