A NON-CHAINABLE PLANE CONTINUUM WITH SPAN ZERO

L. C. HOEHN

Abstract. A plane continuum is constructed which has span zero but is not chainable.

1. Introduction

1.1. Background. The notion of the span of a continuum was introduced by Lelek in [8]. There he proved that chainable continua have span zero, and in 1971 ([9]) he asked whether the converse also holds. This is known as Lelek’s problem, and has become a topic of much interest in continuum theory, in part because there are few other means presently available to decide whether a given continuum is chainable. An affirmative answer to Lelek’s problem would have provided a useful tool with applications to other open problems in continuum theory; for instance, it would have completed the classification of planar homogeneous continua (see [20]).

Lelek’s problem has been featured in a number of recent surveys, appearing as Problem 8 in [5], Problem 2 in [7], Problem 81 in [4], Conjecture 2 in [12], and in [15, p. 255].

There has been previous work toward finding a counterexample for Lelek’s problem. Repovš et al. exhibit in [21] a sequence of trees in the plane with arbitrarily small (positive) spans, none of which has a chain cover of mesh < 1. In [1], Bartošová et al. consider generalizations of the notions of chainability and span zero to the class of Hausdorff (not necessarily metrizable) continua, and prove via a model-theoretic construction that a counterexample for Lelek’s problem in that context would imply that there exists a metric counterexample.

Many positive partial results for Lelek’s problem have been obtained in [13], [16], [17], and [20]. Notably, Minc proves in [13] that span zero is equivalent to chainability among those continua which are inverse limits of trees with simplicial bonding maps, and Oversteegen does the same in [16] for continua which are the image of a chainable continuum under an induced map.

A number of properties of chainable continua have been established for span zero continua. It is known that span zero continua are atriodic [8], and Oversteegen and Tymchatyn show in [19] that they are tree-like and weakly chainable. Further, Marsh proves in [11] that products of span zero continua have the fixed point property, and Bustamante et al. prove in [3] theorems about fixed point and universality properties in the hyperspace of subcontinua of a span zero continuum, generalizing corresponding theorems for chainable continua.

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In this paper, we give an example showing that in general span zero does not imply chainable, even among continua in the plane. This example also provides a negative answer to a question of Mohler (Problem 171 of [4] and Problem 7 of [10]), which asks whether every weakly chainable atriodic tree-like continuum is chainable.

The example presented here is a simple-triod-like continuum, which we will develop as a nested intersection of thickened simple triods in the plane. In Section 2, we introduce some terminology that is useful for describing these simple triods in a combinatorial way. We then show how to extract combinatorial information from a given chain cover of a graph described this way in Section 3 (see [16] for some related work). Section 4 contains the necessary combinatorial lemmas pertaining to our particular graphs, and in Section 5 we construct the example precisely and prove it has the stated properties.

1.2. Definitions and notation. A continuum is a compact connected metric space. We will always denote the metric by \( d \).

Given a continuum \( X \), the span of \( X \) is the supremum of all \( \eta \geq 0 \) for which there exists a subcontinuum \( Z \) of \( X \times X \) such that: 1) \( d(x, y) \geq \eta \) for each \( (x, y) \in Z \); and 2) \( \pi_1(Z) = \pi_2(Z) \), where \( \pi_1, \pi_2 : X \times X \to X \) are the first and second coordinate projections, respectively.

The following facts are straightforward (see [8]):

- if \( X \) and \( Y \) are continua with \( X \subseteq Y \), then \( \text{span}(X) \leq \text{span}(Y) \);
- the arc \([0, 1]\) has span zero; and
- if \( \langle X_n \rangle_{n=1}^\infty \) is a sequence of continua in a given compact metric space, then
  \[
  \limsup_{n \to \infty} \text{span}(X_n) \leq \text{span}(\limsup_{n \to \infty} X_n).
  \]

The third fact implies in particular that given any space \( X \) and any \( \varepsilon > 0 \), there is some \( \delta > 0 \) such that \( \text{span}(X_\delta) < \text{span}(X) + \varepsilon \), where \( X_\delta \) denotes the \( \delta \)-neighborhood of \( X \).

A chain cover of a continuum \( X \) is a finite open cover \( U = \langle U_\ell \rangle_{\ell < L} \) which is enumerated in such a way that \( U_{\ell_1} \cap U_{\ell_2} \neq \emptyset \) if and only if \( |\ell_1 - \ell_2| \leq 1 \). \( X \) is chainable if every open cover of \( X \) has a refinement which is a chain cover.

A simple triod is a continuum \( T \) which is the union of three arcs, \( A_1, A_2, A_3 \), which have a common endpoint \( o \) and are otherwise pairwise disjoint. \( A_1, A_2, A_3 \) are called the legs of \( T \), and \( o \) is the branch point of \( T \).

If \( f : X \to Y \) is a function and \( x_1, \ldots, x_n \in X \), we will often write

\[
x_1 \cdots x_n \mapsto y_1 \cdots y_n
\]

to mean \( f(x_i) = y_i \) for each \( i \).

Given a set \( S \), a total quasi-order on \( S \) is a binary relation \( \leq \) on \( S \) which is reflexive and transitive, and which satisfies the property that for every \( s_1, s_2 \in S \), we have \( s_1 \leq s_2 \) or \( s_2 \leq s_1 \) (or both). If \( \leq \) is a total quasi-order, we write \( s_1 \simeq s_2 \) to mean \( s_1 \leq s_2 \) and \( s_2 \leq s_1 \), and we write \( s_1 < s_2 \) to mean \( s_1 \leq s_2 \) and \( s_2 \not\leq s_1 \).

If \( S \) is finite and \( \leq \) is a total quasi-order on \( S \), then there is a function \( f : S \to \mathbb{Z} \) which is order preserving (i.e. \( f(s_1) \leq f(s_2) \) iff \( s_1 \leq s_2 \)) whose range is a contiguous block of integers.

By a graph, we will mean an undirected connected graph without multiple edges or loops (i.e. edge from a vertex to itself). If \( G \) is a graph, \( V(G) \) denotes the set of
vertices. A pair of vertices \( v_1, v_2 \in V(G) \) are adjacent in \( G \) provided there is an edge between them. A sequence of distinct vertices \( v_1, \ldots, v_n \in V(G) \) are consecutive in \( G \) provided there is an edge between \( v_i \) and \( v_{i+1} \) for each \( 0 \leq i \leq n-1 \).

A graph \( G \) will be considered as a topological space in the usual way, where the edges are realized by arcs. If \( v_1, v_2 \in V(G) \) are adjacent in \( G \), then we will use the notation \([v_1, v_2]\) to denote the arc joining \( v_1 \) and \( v_2 \).

If \( T \) is a tree and \( a, b \in T \), then \([a, b]\) denotes the minimal arc \( A \subseteq T \) with \( a, b \in A \).

By a word, we will mean a finite sequence of symbols. If \( \omega \) is a word, then \( |\omega| \) denotes the length of \( \omega \). A word \( \omega \) will be considered as a function on the set of integers \( \{0, 1, \ldots, |\omega| - 1\} \). \( \omega^{-1} \) denotes the reverse of \( \omega \), defined by \( \omega^{-1}(j) = \omega(|\omega| - 1 - j) \).

Given words \( \omega_1, \omega_2 \) such that the last symbol of \( \omega_1 \) coincides with the first symbol of \( \omega_2 \), define \( \omega_1 \upharpoonright \omega_2 \) to be the word obtained by concatenating onto \( \omega_1 \) all but the first symbol in \( \omega_2 \). For example, \( abc \upharpoonright cab = abcaba \).

2. Graph-words

2.1. Sketches and the graph-word \( \rho_N \).

Definition. A graph-word in the alphabet \( \Gamma \) is a pair \( \rho = (G_\rho, w_\rho) \) where \( G_\rho \) is a graph, and \( w_\rho : V(G_\rho) \to \Gamma \) is a function.

Let us fix, for the rest of this paper, the alphabet \( \Gamma := \{a, b, c\} \cup \{d_t : t \in [0, 1]\} \).

For each positive integer \( N \), denote by \( \alpha_N, \beta_N, \gamma_N \) the following three words:

\[
\begin{align*}
(abc)^{2N+1} \prod_{i=0}^{2N-1} a & a^{2N-i-1} c b (abc)^{2N-i-1} a d_1 c d_1 a (cba)^{2N+1} \\
(abc)^{2N+1} \prod_{i=0}^{2N-1} a & a^{2N-i-1} c b a c (abc)^{2N-i-1} a d_1 c d_1 a (cba)^{2N+1} c b \\
& a c
\end{align*}
\]

For later use, we also define the word \( \beta_N^{-1} \) to be identical to the word \( \beta_N \) except without the final \( b \).

Define the graph-word \( \rho_N \) as follows. Let \( G_{\rho_N} \) be a simple triod, with vertex set \( V(G_{\rho_N}) = \{o, p_1, \ldots, p_{|\alpha_N|-1}, q_1, \ldots, q_{|\beta_N|-1}, r\} \), where \( o \) is the branch point of the triod, \( p_{|\alpha_N|-1}, q_{|\beta_N|-1}, r \) are the endpoints of \( G_{\rho_N} \), the points \( p_j \) belong to the leg \([o, p_{|\alpha_N|-1}]\) with \( p_j \in [o, p_{j+1}] \) for each \( j \), and the points \( q_j \) belong to the leg \([o, q_{|\beta_N|-1}]\) with \( q_j \in [o, q_{j+1}] \) for each \( j \). Put \( p_0 := o \) and \( q_0 := o \). Define \( w_{\rho_N} : V(G_{\rho_N}) \to \Gamma \) by \( w_{\rho_N}(p_1) := \alpha_N(j) \), \( w_{\rho_N}(q_1) := \beta_N(j) \), and \( w_{\rho_N}(r) := \gamma_N(1) = c \).

To construct the example of a non-chainable continuum \( X \) with span zero, we will define a sequence of simple triods \( \langle T_N \rangle_{N=0} \) such that \( T_N \) is contained in a small neighborhood of \( T_{N-1} \) for each \( N > 0 \); \( X \) will then be defined as the intersection of the nested sequence of neighborhoods of the triods \( T_N \). The graph-word \( \rho_N \) will be used to describe the pattern with which we nest the simple triod \( T_N \) inside a small neighborhood of \( T_{N-1} \). To carry this out precisely, we introduce the notion of a sketch below.

Remark. The space \( X \) may alternatively be described as an inverse limit of simple triods, as follows. Let \( T \) be a simple triod with endpoints denoted as \( a, b, c \) and...
branch point \( a \). Denote a point in the interior of the arc \([a, b]\) by \( d_0 \), and parameterize the arc \([d_0, b]\) by \( d_t \) for \( t \in [0, 1] \), so that \( d_1 = b \) (as per the notion of a \( \Gamma \)-marking defined below). Then the \( N \)-th bonding map \( b_N : T \to T \) takes \( a \) to \( a \), is the identity on the segment \([d_0, b]\), and otherwise maps the legs \([a, d_0], [a, b], [a, c]\) in a piecewise linear way according to the patterns \( \alpha_N, \beta_N, \gamma_N \), respectively. Figures 1, 2, and 3, along with the proof of Proposition 1 below, provide some geometric intuition for how this looks.

**Definition.** Given a simple triod \( T \) with branch point \( a \), a \( \Gamma \)-**marking** of \( T \) is a function \( \iota : \Gamma \to T \) such that \( \iota(a), \iota(b), \iota(c) \) are the endpoints of \( T \) and \( \{\iota(d_t) : t \in [0, 1]\} \subset [a, \iota(b)] \) are such that whenever \( t < t' \), we have \( \iota(d_t) \in [\iota(d_t'), \iota(d_{t'})] \) and \( \text{diam}(\iota(d_t), \iota(d_{t'})) = d(\iota(d_t), \iota(d_{t'})) = t' - t \).

Define the simple triod \( T_0 := \{(x, 0) : x \in [-1, 1]\} \cup \{(0, y) : y \in [0, 2]\} \subset \mathbb{R}^2 \), and define a \( \Gamma \)-marking \( \iota : \Gamma \to T_0 \) by:

\[
\iota(a) := (-1, 0) \\
\iota(b) := (0, 2) \\
\iota(c) := (1, 0) \\
\iota(d_t) := (0, 1 + t) \quad \text{for} \ t \in [0, 1].
\]

To simplify definitions and arguments in the following, we will restrict our attention to a special class of graph-words.

**Definition.** A **compliant graph-word** is a graph-word \( (G, w) \) in the alphabet \( \Gamma \) such that there is no pair of adjacent vertices \( v_1, v_2 \) in \( G \) with \( w(v_1) \approx_{T} w(v_2) \).

Observe that \( \rho_N \) is a compliant graph-word for each \( N \).

**Definition.** Suppose \( T \) is a simple triod with a \( \Gamma \)-marking \( \iota : \Gamma \to T \), and let \( \rho = (G, w) \) be a compliant graph-word in the alphabet \( \Gamma \). Then \( \tilde{w} : G \to T \) is a \( \rho \)-**suggested bonding map** provided \( \tilde{w}|_{V(G)} = \iota \circ w \), and for any adjacent \( v_1, v_2 \in V(G) \), we have that \( \tilde{w}|_{[v_1, v_2]} \) is a homeomorphism from \([v_1, v_2]\) to \([\iota(w(v_1)), \iota(w(v_2))]\).

**Definition.** Let \( (\Omega, d) \) be a metric space, let \( T \subset \subset \Omega \) be a \( \Gamma \)-marked simple triod, let \( G \subset \subset \Omega \) be a graph, and let \( \varepsilon > 0 \). Then \( \rho = (G, w) \) is a \( (T, \varepsilon) \)-**sketch of \( G \) in \( \Omega \)** if \( \rho \) is a compliant graph-word in the alphabet \( \Gamma \), and there is a \( \rho \)-suggested bonding map \( \tilde{w} : G \to T \) such that \( d(x, \tilde{w}(x)) < \frac{\varepsilon}{2} \) for every \( x \in G \).

The next proposition assures us that we may use the graph word \( \rho_N \) defined above to describe the pattern with which we embed one simple triod into a small neighborhood of another, in the plane.

We will need some additional notation when working with the graph-word \( \rho_N \). For each \( i \leq 2N \), define \( n(i) \) and \( m(i) \) to be the unique integers such that

\[
(n(i) - 1) \cdot n(i) \cdot (n(i) + 1) \overset{\alpha_S}{\Rightarrow} d_{i/2N} \circ d_{i/2N} \\
(m(i) - 1) \cdot m(i) \cdot (m(i) + 1) \overset{\beta_S}{\Rightarrow} d_{i/2N} \circ d_{i/2N}.
\]

For each \( i < 2N \), define \( \theta(i) := 6N - 3i + 1 \) and \( \phi(i) := 6N - 3i + 2 \), so that

\[
(n(i) + \theta(i) - 1) \cdot (n(i) + \theta(i)) \cdot (n(i) + \theta(i) + 1) \overset{\alpha_S}{\Rightarrow} cbc \\
(m(i) + \phi(i) - 2) \cdot (m(i) + \phi(i)) \cdot (m(i) + \phi(i) + 2) \overset{\beta_S}{\Rightarrow} chabc.
\]
Note that \( n(0) = m(0) = 6N + 5 \), and that \( n(i) + 2\theta(i) = n(i + 1) \) and \( m(i) + 2\phi(i) = m(i + 1) \) for each \( i < 2N \).

**Proposition 1.** Suppose \( T \subset \mathbb{R}^2 \) is a simple triod and \( \iota : \Gamma \to T \) is a \( \Gamma \)-marking. For each integer \( N > 0 \) and any \( \varepsilon > 0 \), there is an embedding of the simple triod graph \( G_{\rho N} \) in \( \mathbb{R}^2 \) such that \( \rho_N \) is a \( (T, \varepsilon) \)-sketch of \( G_{\rho N} \) in \( \mathbb{R}^2 \). Moreover, the embedding can be chosen to satisfy \( \lvert q_i |_{\beta_{N}^{-2}}, q_i |_{\beta_{N}^{-1}} \rvert = \lvert \iota(c), \iota(b) \rvert \).

Observe that this proposition would be more or less immediate if we were to replace \( \mathbb{R}^2 \) by \( \mathbb{R}^3 \). Thus, the reader who is content with a non-planar counterexample for Lelek’s problem may skip the details.

**Proof.** For simplicity, we will argue only the case \( T = T_0 \), with the \( \Gamma \)-marking \( \iota \) as described above; the general case can be treated similarly.

First we will analytically define a different embedding of \( G_{\rho_N} \) in \( \mathbb{R}^2 \), then we will describe how to obtain the desired embedding from it.

Let \( \eta > 0 \) be significantly smaller than \( \varepsilon \), say \( \eta < \frac{\varepsilon}{40N^2} \). For \( 0 \leq i \leq 2N \), put
\[
\begin{align*}
p_n(i) & := (1 + \eta, (4i + \frac{3}{2})\eta), \\
qu_m(i) & := (1, (4i + \frac{3}{2})\eta).
\end{align*}
\]

For \( 0 \leq i < 2N \) and \( 1 \leq j < \theta(i) \), put
\[
\begin{align*}
p_n(i)_{+j} & := (1 - j, (4i + 3)\eta), \\
p_n(i+1)_{-j} & := (1 - j, 4(i + 1)\eta),
\end{align*}
\]
and put \( p_n(i)_{+\theta(i)} := (1 - \theta(i), (4i + \frac{7}{2})\eta) \). For \( 0 \leq i < 2N \) and \( 1 \leq j < \phi(i) \), put
\[
\begin{align*}
q_m(i)_{+j} & := (1 - j, (4i + 2)\eta), \\
q_m(i+1)_{-j} & := (1 - j, (4(i + 1) + 1)\eta),
\end{align*}
\]
and put \( q_m(i)_{+\phi(i)} := (1 - \phi(i), (4i + \frac{7}{2})\eta) \). Further, put
\[
\begin{align*}
p_n(0)_{-j} & := (1 - j, 0) & \text{ for } 1 \leq j \leq 6N + 5, \\
qu_m(0)_{-j} & := (1 - j, \eta), & \text{ for } 1 \leq j \leq 6N + 5, \\
qu_m(2N)_{+j} & := (1 - j, (8N + 2)\eta), & \text{ for } 1 \leq j \leq 6N + 7, \\
p_n(2N)_{+j} & := (1 - j, (8N + 3)\eta), & \text{ for } 1 \leq j \leq 6N + 5.
\end{align*}
\]

Finally, put \( o := (-6N - 4, \frac{1}{2}\eta) \) and \( r := (-6N - 5, \frac{1}{2}\eta) \). Join each pair of adjacent vertices in \( G_{\rho_N} \) by a straight line segment in \( \mathbb{R}^2 \). Denote the resultant embedding of \( G_{\rho_N} \) in \( \mathbb{R}^2 \) by \( G' \). Figure 1 depicts the embedding \( G' \) for \( N = 1 \).

Observe that in \( G' \), for each integer \( k \leq -1 \), if \( v \) and \( v' \) are two vertices in the line \( x = k \), then \( w(v) = w(v') \). Also notice that each vertex \( v \) in the line \( x = -1 \) is already close to the point \( \iota(w(v)) = \iota(a) = (-1, 0) \), and that each vertex \( u \) of the form \( p_n(i) \) or \( q_m(i) \) is already close to the point \( \iota(w(u)) = \iota(c) = (1, 0) \). We now describe heuristically in two steps how to mold \( G' \) into the embedding we seek.

First, for each \( i \leq 2N \), for each triple \( \langle v_1, v_2, v_3 \rangle \) of the form \( \langle p_n(i)_{-2}, p_n(i-1)_{-1}, p_n(i) \rangle \), \( \langle q_m(i)_{-2}, q_m(i-1), q_m(i) \rangle \), \( \langle p_n(i)_{+2}, p_n(i+1)_{+1}, p_n(i) \rangle \), or \( \langle q_m(i)_{+2}, q_m(i+1)_{+1}, q_m(i) \rangle \), move the vertex \( v_2 \) up to be close to the point \( \iota(d_{i/2N}) \), move the vertex \( v_3 \) down slightly, and shape the arcs joining \( v_1 \) to \( v_2 \) and \( v_2 \) to \( v_3 \) so that:

1. there is a homeomorphism \( \hat{w}_1 : [v_1, v_2] \to [\iota(a), \iota(d_{i/2N})] \subset T_0 \) such that \( \hat{w}_1(v_1) = \iota(a), \hat{w}_1(v_2) = \iota(d_{i/2N}) \), and \( d(x, \hat{w}_1(x)) < \eta \) for each \( x \in [v_1, v_2] \),
(2) there is a homeomorphism \( \tilde{w}_2 : [v_2, v_3] \rightarrow [\iota(d_{i/2N}), \iota(c)] \subset T_0 \) such that
\[
\tilde{w}_2(v_2) = \iota(d_{i/2N}), \quad \tilde{w}_2(v_3) = \iota(c), \quad \text{and} \quad d(x, \tilde{w}_2(x)) < \eta \text{ for each } x \in [v_2, v_3],
\]
and so that in the end no new intersections between those arcs have been introduced
(i.e., so that the result is still an embedding of \( G_{\rho N} \)). Figure 2 depicts the result
for \( N = 1 \).
Next, take the strip \( \{(x, y) : x \leq -1, 0 \leq y \leq (8N + 3)\eta\} \) and stretch and wind it counter-clockwise \( 2N + 2 \) times around the outside of

\[
\bigcup_{i=0}^{2N} \left( [p_n(i) - 2, p_n(i) + 2] \cup [q_m(i) - 2, q_m(i) + 2] \right),
\]

so that for each integer \( k \leq -1 \), all the vertices \( v \) in the line \( x = k \) end up near the point \( \iota(w(v)) \in T_n \), taking care to make sure \( [q_{|\beta_N| - 2}, q_{|\beta_N| - 1}] = [\iota(c), \iota(b)] \). Figure 3 depicts roughly how this wrapping looks.

The resulting embedding satisfies the desired properties. \( \square \)

2.2. Span and \( \rho_N \). In this section we prove that the span of a simple triod described by \( \rho_N \) converges to 0 as \( N \to \infty \). This ensures that we will obtain a
continuum with span zero when we take the nested intersection of neighborhoods of triods described by the \( \rho_N \)'s.

**Lemma 2.** Let \( T \) be a simple triod with legs \( A_1, A_2, A_3 \) and branch point \( o \). For each \( i \) let \( p_i \) be the endpoint of leg \( A_i \) other than \( o \). Suppose \( \delta > 0 \) and \( W \subset A_1 \times A_2 \) is an arc such that \((o,o) \in W, W \) meets \( \{(p_1) \times A_2) \cup (A_1 \times \{p_2) \}) \), and \( d(x_1,x_2) \leq \delta \) for each \((x_1,x_2) \in W \). Then the span of \( T \) is \( \leq \delta \).

**Proof.** Suppose \( Z \subset T \times T \) is a subcontinuum with \( \pi_1(Z) = \pi_2(Z) \). If \( \pi_1(Z) \) is an arc, then it is easy to see that \( Z \) meets the diagonal \( \Delta T = \{(x,x) : x \in T\} \), as arcs have span zero.

If \( \pi_1(Z) \) is a subtrioid \( T' \) of \( T \), then we may assume \( T = T' \) by replacing the arc \( W \) by the component of \( W \cap (T' \times T') \) that contains \((o,o) \). Let \( K_1 \) and \( K_2 \) be disjoint clopen subsets of \((A_1 \times A_2) \setminus W \) such that \((A_1 \times \{o\}) \setminus W \subset K_1 \), \((\{o\} \times A_2) \setminus W \subset K_2 \), and \( K_1 \cup K_2 = (A_1 \times A_2) \setminus W \).

For each \( i \in \{1,2,3\} \) let \( U_i \) and \( V_i \) be the two components of \((A_i \times A_i) \setminus \Delta T \), where \((A_i \setminus \{o\}) \times \{o\} \subset U_i \) and \( \{o\} \times (A_i \setminus \{o\}) \subset V_i \). It can then be seen that the set

\[
Y := (U_1 \cup U_2 \cup V_3) \cup (A_1 \times A_3) \cup (A_2 \times A_3) \cup K_1 \cup K_2^{-1} \setminus W
\]

is clopen in \((T \times T) \setminus (W \cup W^{-1} \cup \Delta T) \) (see Proposition 5.1 of [6]).

Observe that \( p_i \notin \pi_1(Y) \) and \( p_i \notin \pi_2((T \times T) \setminus Y) \), hence \( Z \notin Y \) and \( Z \notin (T \times T) \setminus Y \). Since \( Z \) is connected, it follows that \( Z \) must meet \( W \cup W^{-1} \cup \Delta T \).

Thus in either case, there is some \((x_1,x_2) \in Z \) with \( d(x_1,x_2) \leq \delta \). Therefore \( T \) has span \( \leq \delta \).

**\( \square \)**

**Proposition 3.** Suppose \( T \subset R^2 \) is \( \Gamma \)-marked. If the triod graph \( G_{\rho_N} \) is embedded in \( R^2 \) such that \( \rho_N \) is a \( (T,\epsilon) \)-sketch of \( G_{\rho_N} \) in \( R^2 \), then the span of \( G_{\rho_N} \) is less than \( \frac{1}{2N} + \epsilon \).

**Proof.** In order to apply Lemma 2, we will produce an arc \( W \subset [o,p_{[\alpha_N]-1}] \times [o,q_{[\beta_N]-1}] \). Intuitively, one may think of \( W \) as a pair of points travelling simultaneously, one on the leg \([o,p_{[\alpha_N]-1}] \) and the other on \([o,q_{[\beta_N]-1}] \), starting with both at \( o \) and ending with one at the end of its leg, and at every moment staying within distance \( \frac{1}{2N} + \epsilon \) from one another. With this in mind, and referring to Figure 1, one should be easily convinced that such a \( W \) may be defined which passes through the following pairs, in order: \((o,o),(p_{n(0)},q_{m(0)}),(p_{n(0)+\phi(0)},q_{m(0)+\phi(0)}),(p_{n(0)},q_{m(1)}),(p_{n(0)+\theta(0)},q_{m(1)-\theta(0)}),(p_{n(1)},q_{m(1)}),\ldots,(p_{n(2N)},q_{m(2N)}),(p_{2N-1},q_{m(2N)+6N+5})\).

The precise definition of this arc \( W \) follows.

Suppose that \( n, n' \) and \( m, m' \) are two pairs of adjacent integers. Let \( S_{m,m'}^{n,n'} \) denote the square \([p_n,p_{n'}) \times [q_m,q_{m'}] \). Suppose one of the following occurs:

1. \( w(p_n) = w(q_n), w(p_{n'}) = w(q_{m'}) \);
2. \( w(p_n) = w(q_n), w(p_{n'}) = d_{i/N}, w(q_{m'}) = d_{i+1/N} \) for some \( i \); or
3. \( w(p_{n'}) = w(q_{m'}), w(p_n) = d_{i/N}, w(q_n) = d_{i+1/N} \) for some \( i \).

Then let \( W_{m,m'}^{n,n'} \subset S_{m,m'}^{n,n'} \) be an arc such that \( d(x_1,x_2) < \frac{1}{2N} + \epsilon \) for each \((x_1,x_2) \in W_{m,m'}^{n,n'} \), and \( W_{m,m'} \cap \partial S_{m,m'}^{n,n'} = \{(p_n,q_m), (p_{n'},q_{m'})\} \).
Define the arc \( W \subset [o, p_{[0,N]}-1] \times [o, q_{[0,N]}-1] \) as follows. It will be helpful to refer to Figure 1 when reading this formula.

\[
W := \bigcup_{j=0}^{n(0)-1} W_{j,j+1}^{n(j)-1} \cup \bigcup_{i=0}^{2N-1} \left( \bigcup_{j=0}^{\phi(i)-1} W_{j,j+1}^{n(i)-j, n(i)-j-1} \cup \bigcup_{j=0}^{\theta(i)-1} W_{m(i)+j, m(i)+j+1}^{n(i)+j, n(i)+j+1} \right) \cup \bigcup_{j=0}^{6N+4} W_{m(2N)+j, m(2N)+j+1}^{n(2N)+j, n(2N)+j+1}.
\]

Then \( W \) contains \( (o, o) \) and meets \( \{p_{[0,N]}-1\} \times [o, q_{[0,N]}-1] \), and \( d(x_1, x_2) < \frac{1}{2N} + \varepsilon \) for each \( (x_1, x_2) \in W \), hence the claim follows by Lemma 2.

\[\square\]

3. Combinatorics from chain covers


**Definition.** Define the equivalence relation \( \approx_T \) on \( \Gamma \) by \( \sigma \approx_T \tau \) if and only if \( \sigma = \tau \) or \( \sigma, \tau \in \{b\} \cup \{d_t : t \in [0, 1]\} \).

The relation \( \approx_T \) partitions \( \Gamma \) into three equivalence classes. If \( t \) is a \( \Gamma \)-marking of a triod \( T \), then \( \sigma \approx_T \tau \) if and only if \( \iota(\sigma) \) and \( \iota(\tau) \) belong to the same leg of \( T \).

The following definition is closely related to the notion of a chain word reduction from [14]. It should be thought of as follows: if \( \langle G, w \rangle \) is a \( \langle T, \varepsilon \rangle \)-sketch of \( G \) and we have a chain cover of \( G \) of small mesh, then \( v_1 \leq v_2 \) means roughly that the chain “covers \( v_1 \) before, or at around the same time as, \( v_2 \)” (see Proposition 5).

**Definition.** Suppose \( \langle G, w \rangle \) is a compliant graph-word. A chain quasi-order of \( \langle G, w \rangle \) is a total quasi-order \( \preceq \) on \( V(G) \) satisfying:

- **(C1)** if \( v_1 \preceq v_2 \), then \( w(v_1) \approx_T w(v_2) \);
- **(C2)** if \( v_1, v_2 \in V(G) \) are adjacent in \( G \), then \( v_1 \) and \( v_2 \) are \( \leq \)-adjacent; and
- **(C3)** if \( v_1, v_2, v_3 \in V(G) \) are consecutive in \( G \), \( v \in V(G) \), and if \( \sigma, \tau \in \{a, c\} \) and \( t, t' \in [0, 1] \) are such that \( t' \geq t \), \( v_1 v_2 v_3 \overset{w}{\rightarrow} \sigma d_t \tau \), \( w(v) = d_t \), and \( v_1 < v_2 \preceq v < v_3 \), then \( t' - t < \frac{1}{2} \).

Notice that if \( \preceq \) is a chain quasi-order, then the reverse order of \( \leq \) (defined by \( v_1 \preceq v_2 \text{ iff } v_2 \preceq v_1 \)) is also a chain quasi-order.

The following simple lemma will be useful later on.

**Lemma 4.** Let \( \preceq \) be a chain quasi-order of \( \langle G, w \rangle \). Suppose \( v_1, s_1, \ldots, s_\kappa, v_2 \) are consecutive in \( G \) and \( v \in V(G) \) is such that \( v_1 < v < v_2 \). Then there is some \( i \in \{1, \ldots, \kappa\} \) such that \( v \preceq s_i \).
Proof. Put $s_0 := v_1$, $s_{i+1} := v_2$, and let $i$ be the largest integer in $\{0, \ldots, \kappa\}$ such that $s_i \leq v$. Then $s_{i+1} > v$, so since $s_i$ and $s_{i+1}$ are $\leq$-adjacent by property (C2), we must have $s_i \geq v$. Thus $s \equiv v$.

3.2. Chain covers and the triod $T_0$.

Proposition 5. Suppose $(G, w)$ is a compliant graph-word which is a $(T_0, \varepsilon)$-sketch of a graph $G$ in $\mathbb{R}^2$. If there is a chain cover for $G$ of mesh $< \frac{1}{2} - \varepsilon$, then there is a chain quasi-order of $(G, w)$.

Proof. Let $\mathcal{U} = \{U_\ell : \ell < L\}$ be a chain cover for $G$ of mesh $< \frac{1}{2} - \varepsilon$, ordered so that $U_\ell \cap U_{\ell'} \neq \emptyset$ iff $|\ell - \ell'| \leq 1$. For each $v \in V(G)$, let $\ell(v)$ be such that $v \in U_{\ell(v)}$ (for each $v$ there are either one or two choices for $\ell(v)$).

Observe that if $v_1, v_2 \in V(G)$ and $\ell(v_1) = \ell(v_2)$, then $w(v_1) \approx_G w(v_2)$, since otherwise $d(i(w(v_1)), i(w(v_2))) \geq \sqrt{2} > \frac{1}{2}$, hence $d(v_1, v_2) > \frac{1}{2} - \varepsilon$, contradicting the fact that the diameter of $U_{\ell(v_1)} = U_{\ell(v_2)}$ is $< \frac{1}{2} - \varepsilon$.

Define the relation $\leq$ on $V(G)$ by setting $v_1 \leq v_2$ if and only if for every $v \in V(G)$ satisfying $\ell(v_2) \leq \ell(v) \leq \ell(v_1)$ we have $w(v) \approx_G w(v_1)$.

The following facts follow directly from the definition of $\leq$:

**Facts.**

1. If $\ell(v_1) \leq \ell(v_2)$, then $v_1 \leq v_2$.
2. If $v_1 \leq v_2$ and $w(v_1) \not\approx_G w(v_2)$, then $\ell(v_1) < \ell(v_2)$.
3. If $v_1, v_2 \in V(G)$ are $\leq$-adjacent, then $w(v_1) \not\approx_G w(v_2)$.

It is straightforward to check using the definition and these facts that $\leq$ is a total quasi-order.

We now check that $\leq$ satisfies properties (C1), (C2), and (C3) of the definition of a chain quasi-order.

(C1): Suppose $v_1, v_2 \in V(G)$ with $v_1 \simeq v_2$. Assume without loss of generality that $\ell(v_2) \leq \ell(v_1)$. It then follows immediately from the definition of $\leq$ and the assumption $v_1 \leq v_2$ that $w(v_1) \approx_G w(v_2)$ (take $v = v_2$).

(C2): Suppose $v_1, v_2 \in V(G)$ are adjacent in $G$. Since $(G, w)$ is compliant, we know that $w(v_1) \not\approx_G w(v_2)$. Let $\sigma := w(v_1)$ and $\tau := w(v_2)$. Assume without loss of generality that $\ell(v_1) < \ell(v_2)$, which implies that $v_1 < v_2$.

If $v \in V(G)$ were such that $w(v) \not\approx_G \sigma, \tau$ and $v_1 < v < v_2$, then $\ell(v_1) < \ell(v) < \ell(v_2)$. This would imply that the link $U_{\ell(v)}$ contains the point $v$ and meets the arc $[v_1, v_2]$. Since $(G, w)$ is compliant, the only possible cases are:

$$\{\sigma, \tau\} = \{a, b\} \text{ and } w(v) = c$$
$$\{\sigma, \tau\} = \{a, c\} \text{ and } w(v) \in \{b\} \cup \{d_t : t \in [0, 1]\}$$
$$\{\sigma, \tau\} = \{b, c\} \text{ and } w(v) = a$$
$$\{\sigma, \tau\} = \{a, d_t\} \text{ and } w(v) = c \quad (\text{for some } t \in [0, 1])$$
$$\{\sigma, \tau\} = \{c, d_t\} \text{ and } w(v) = a \quad (\text{for some } t \in [0, 1])$$

In each case, we have $d(i(w(v))), [i(\sigma), i(\tau))] \geq 1 > \frac{1}{2}$. But this yields a contradic-

tion, since $\mathcal{U}$ has mesh $< \frac{1}{2} - \varepsilon$.

Suppose for a contradiction that $v_1, v_2$ are not adjacent in the $\leq$ order. Let $v, v'$ be such that $v_1 < v < v'$, and $v_1, v$ are $\leq$-adjacent and $v, v'$ are $\leq$-adjacent. By the above, we have that $w(v), w(v')$ are each $\approx_G$ to either $\sigma$ or $\tau$, hence by
Fact (3) the only possibility is \( w(v) \approx_\tau \tau, w(v') \approx_\sigma \sigma \). Fact (2) then implies that \( \ell(v_1) < \ell(v) < \ell(v') \).

Define the arc \( A \subset T_0 \) according to the value of \( \sigma \) as follows:

\[
A := \begin{cases} 
\iota(a, o) & \text{if } \sigma = a \\
\iota(b, o) & \text{if } \sigma = c \\
\iota(c, o) & \text{if } \sigma = a \\
\{d_t : t \in [0, 1]\} & \text{if } \sigma \in \{b\} \cup \{d_t : t \in [0, 1]\}
\end{cases}
\]

In each case, observe that \( d(\iota(w(v)), A) \geq 1 + \frac{1}{2} \), and also \( B_A(\iota(\sigma)) \subset A \) and \( B_A(\iota(w(v'))) \subset A \).

Applying the homeomorphism \( \bar{w}|_{[v_1, v_2]} \) yields the chain cover \( \bar{w}(U_\ell \cap [v_1, v_2]) : \ell' \leq \ell \leq \ell'' \) of the arc \( [\iota(\sigma), \iota(\tau)] \) in \( T_0 \), where \( \ell' := \min\{\ell : U_\ell \cap [v_1, v_2] \neq \emptyset\} \) and \( \ell'' := \max\{\ell : U_\ell \cap [v_1, v_2] \neq \emptyset\} \).

Notice that \( \bar{w}(U_{\ell(v_1)}) \) and \( \bar{w}(U_{\ell(v')}) \) are sets of diameter \( < \frac{1}{2} \) containing \( \iota(\sigma) \) and \( \iota(w(v')) \), respectively, hence are subsets of \( A \). It follows in particular that the links \( \bar{w}(U_{\ell(v_1)} \cap [v_1, v_2]) \) and \( \bar{w}(U_{\ell(v')} \cap [v_1, v_2]) \) both meet the arc \( A \cap [\iota(\sigma), \iota(\tau)] \), which implies each link \( \bar{w}(U_\ell \cap [v_1, v_2]) \), \( \ell(v_1) < \ell < \ell(v') \), must meet \( A \) as well.

But \( \bar{w}(U_{\ell(v_1)}) \) has diameter \( < \frac{1}{2} \) and contains \( \iota(w(v)) \), hence misses \( A \) by the above. This is a contradiction, therefore we must have that \( v_1 \) and \( v_2 \) are \( \leq \)-adjacent.

(C3): Suppose \( v \in V(G), v_1, v_2, v_3 \) are consecutive in \( G \), and that \( \sigma, \tau \in \{a, c\} \) and \( t, t' \in [0, 1] \) are such that \( t' \geq t \), \( w(v) = d_\sigma \), \( v_1 v_2 v_3 \rightarrow v_3 \sigma d_\tau \), and \( v_1 < v_2 \approx v < v_3 \).

From Fact (2) we know that \( \ell(v) \) is between \( \ell(v_1) \) and \( \ell(v_3) \), hence the link \( U_{\ell(v)} \) contains \( v \) and meets the arc \( [v_1, v_2] \cup [v_2, v_3] \). Since \( d(\iota(d_\sigma)), [\iota(\sigma), \iota(d_\sigma)] \) \( \Rightarrow [\iota(d_\sigma), \iota(\tau)] = d(\iota(d_\sigma), \iota(d_\sigma)) = t' - t \) and \( U \) has mesh \( < \frac{1}{2} - \varepsilon \), it follows that \( t' - t < \frac{1}{2} \).

\[
\square
\]

4. Combinatorics of the graph-word \( \rho_N \)

4.1. Chain quasi-orders and \( \rho_N \). Throughout this subsection assume that \( \langle G, w \rangle \) is a compliant graph-word, and that \( \leq \) is a chain quasi-order of \( \langle G, w \rangle \).

Let \( f : V(G) \rightarrow \mathbb{Z} \) be an order preserving function whose range is a contiguous block of integers.

**Lemma 6.** Suppose \( v_1, \ldots, v_n \) are consecutive in \( G \), and that for each \( 1 \leq j < n \) we have \( w(v_{j-1}) \neq_\Gamma w(v_{j+1}) \). Then \( f(v_1), \ldots, f(v_n) \) are consecutive integers, i.e. either \( f(v_{j+1}) = f(v_j) + 1 \) for each \( 1 \leq j < n \), or \( f(v_{j+1}) = f(v_j) - 1 \) for each \( 1 \leq j < n \).

**Proof.** This follows immediately from properties (C1) and (C2) of the chain quasi-order \( \leq \).

As an application of Lemma 6, we make the following observation.

**Lemma 7.** Suppose for some \( i < 2N \) that \( v_0, v_1, \ldots, v_{2\theta(i)} \in V(G) \) are consecutive in \( G \) with \( v_0 \cdots v_{2\theta(i)} \overset{w}{\rightarrow} \alpha_N(n(i)) \cdots \alpha_N(N(i + 1)) \). Let \( k := f(v_0) \). Then we have one of the following four cases:

1. \( v_0 \cdots v_{2\theta(i)} \overset{f}{\rightarrow} k \cdots (k + 2\theta(i)) \);
2. \( v_0 \cdots v_{\theta(i)} \overset{f}{\rightarrow} k \cdots (k + \theta(i)) \), \( v_{\theta(i)} \cdots v_{2\theta(i)} \overset{f}{\rightarrow} (k + \theta(i)) \cdots k \);
3. \( v_0 \cdots v_{\theta(i)} \overset{f}{\rightarrow} k \cdots (k - \theta(i)) \), \( v_{\theta(i)} \cdots v_{2\theta(i)} \overset{f}{\rightarrow} (k - \theta(i)) \cdots k \); or
Lemma 6. For each \(i\)

Suppose for a contradiction that

\[ \text{Proof.} \]

It then follows from the pigeonhole principle that

\[ \text{Suppose} \]

Lemma 9.

\[ \text{Proof.} \]

Suppose for a contradiction that \(f(v_3^{(0)}) = f(v_3^{(N)}) = f(v_3^{(2N)}) = k\). By Lemma 6, for each \(i \in \{0, N, 2N\}\) we have either

\[ v_1^{(i)} v_2^{(i)} v_3^{(i)} \mapsto (k - 2)(k - 1)k \]

or

\[ v_1^{(i)} v_2^{(i)} v_3^{(i)} \mapsto (k + 2)(k + 1)k. \]

It then follows from the pigeonhole principle that \(f(v_2^{(i)}) = f(v_2^{(j)})\) for distinct \(i, j \in \{0, N, 2N\}\). But this contradicts property (C3) of the chain quasi-order \(\leq\).

Lemma 9. Suppose \(v_0, \ldots, v_{|\alpha_N|-1} \in V(G)\) are consecutive in \(G\) and \(v_0', \ldots, v_{|\beta_N|-2} \in V(G)\) are consecutive in \(G\) with \(v_0 \cdots v_{|\alpha_N|-1} \mapsto \alpha_N\) and \(v_0' \cdots v_{|\beta_N|-2} \mapsto \beta_N\). Suppose further that \(v_0 \preceq v_0'\). Then \(v_1 \not\sim v_1'\).

\[ \text{Proof.} \]

Assume without loss of generality that \(v_0 \leq v_1\). Suppose for a contradiction that \(v_1 \preceq v_1'\).

We know that \(f(v_0) \preceq f(v_1)\) and that \(f(v_0) = f(v_0')\), \(f(v_1) = f(v_1')\). Put \(k := f(v_{n(0)})\), and recall that \(n(0) = 6N + 5 = m(0)\). It follows from Lemma 6 that

\[ v_0 \cdots v_{n(0)} \mapsto (k - 6N - 5) \cdots k, \]

\[ v_0' \cdots v_{m(0)}' \mapsto (k - 6N - 5) \cdots k. \]

Claim 9.1. Let \(i < 2N\). If \(f(v_{n(i)}) = k\) and \(f(v_{n(i)} + \theta(i)) < k\), then \(f(v_{n(i+1)}) = k\). Similarly, if \(f(v'_{m(i)}) = k\) and \(f(v'_{m(i)} + \phi(i)) < k\), then \(f(v'_{m(i+1)}) = k\).

\[ \text{Proof of Claim 9.1.} \]

Suppose \(f(v_{n(i)}) = k > f(v_{n(i)} + \theta(i))\). If

\[ v_{n(i)} \cdots v_{n(i+1)} \mapsto (k - 2\theta(i)), \]

then in particular \(f(v_{n(i)} + \theta(i) + 1) = k - \theta(i) - 1\). Also, \(f(v_{n(0)} - \theta(i) - 1) = k - \theta(i) - 1\). But \(w(v_{n(i)} + \theta(i) + 1) = c \equiv r a = w(v_{n(0)} - \theta(i) - 1)\), so this contradicts property (C1) of the chain quasi-order \(\leq\). Therefore by Lemma 7 we must have \(f(v_{n(i+1)}) = k\).

Similarly, suppose \(f(v'_{m(i)}) = k > f(v'_{m(i)} + \phi(i))\). If

\[ v'_{m(i)} \cdots v'_{m(i+1)} \mapsto (k - 2\phi(i)), \]

then in particular \(f(v'_{m(i)} + \phi(i) + 1) = k - \phi(i) - 1\). Also, \(f(v'_{m(0)} - \phi(i) - 1) = k - \phi(i) - 1\). But \(w(v'_{m(i)} + \phi(i) + 1) = b \equiv r c = w(v'_{m(0)} - \phi(i) - 1)\), so this contradicts property (C1) of the chain quasi-order \(\leq\). Therefore by Lemma 7 we must have \(f(v'_{m(i+1)}) = k\).\(\square\)(Claim 9.1)
Claim 9.2. Either \( f(v_{n(i)}) = k \) for each \( i \leq 2N \) or \( f(v'_{m(i)}) = k \) for each \( i \leq 2N \).

Proof of Claim 9.2. If \( f(v_{n(i)+\theta(i)}) < k \) and \( f(v'_{m(i)+\phi(i)}) < k \) for each \( i < 2N \), then this follows immediately from Claim 9.1 and induction. Hence assume this is not the case, and let \( i^* \) be the smallest \( i \) for which \( f(v_{n(i)+\theta(i)}) > k \) or \( f(v'_{m(i)+\phi(i)}) > k \).

Observe that by Claim 9.1 and induction, we have \( f(v_{n(i)}) = f(v'_{m(i)}) = k \) for each \( i \leq i^* \).

Case 1. \( f(v'_{m(i^*)+\phi(i^*)}) > k \).

It follows from Lemma 6 that
\[
v'_{m(i^*)} \cdots v'_{m(i^*)+\phi(i^*)} \xrightarrow{f} k \cdots (k + \phi(i^*)).\]

Suppose \( i^* \leq i < 2N \), and that \( f(v_{n(i)}) = k \). If \( f(v_{n(i)+\theta(i)}) < k \), then we have by Claim 9.1 that \( f(v_{n(i+1)}) = k \).

If \( f(v_{n(i)+\theta(i)}) > k \), suppose for a contradiction that
\[
v_{n(i)} \cdots v_{n(i+1)} \xrightarrow{f} k \cdots (k + 2\theta(i)).\]

In particular, this means \( f(v_{n(i)+\theta(i)+1}) = k + \theta(i) + 1 \). Also, since \( \phi(i^*) > \theta(i) \), we have \( f(v'_{m(i^*)+\theta(i)+1}) = k + \theta(i) + 1 \). But \( w(v_{n(i)+\theta(i)+1}) = c \not\leq \Gamma a = w(v'_{m(i^*)+\theta(i)+1}) \), so this contradicts property (C1) of the chain quasi-order \( \leq \).

Therefore by Lemma 7 we must have \( f(v_{n(i+1)}) = k \).

Thus by induction, we have \( f(v_{n(i)}) = k \) for each \( i \leq 2N \).

Case 2. \( f(v'_{m(i^*)+\phi(i^*)}) < k \) and \( f(v_{n(i^*)+\theta(i^*)}) > k \).

Here we have by Claim 9.1 that \( f(v'_{m(i^*)}) = k \).

It follows from Lemma 6 that
\[
v_{n(i^*)} \cdots v_{n(i^*)+\theta(i^*)} \xrightarrow{f} k \cdots (k + \theta(i^*)).\]

Suppose \( i^* + 1 \leq i < 2N \), and that \( f(v'_{m(i)}) = k \). If \( f(v'_{m(i)+\phi(i)}) < k \), then we have by Claim 9.1 that \( f(v_{m(i+1)}) = k \).

If \( f(v'_{m(i)+\phi(i)}) > k \), suppose for a contradiction that
\[
v'_{m(i)} \cdots v'_{m(i+1)} \xrightarrow{f} k \cdots (k + 2\phi(i)).\]

In particular, this means \( f(v'_{m(i)+\phi(i)+1}) = k + \phi(i) + 1 \). Also, since \( \theta(i^*) > \phi(i) \), we have \( f(v_{n(i^*)+\phi(i)+1}) = k + \phi(i) + 1 \). But \( w(v'_{m(i)+\phi(i)+1}) = b \not\leq \Gamma c = w(v_{n(i^*)+\phi(i)+1}) \), so this contradicts property (C1) of the chain quasi-order \( \leq \).

Therefore by Lemma 7 we must have \( f(v'_{m(i+1)}) = k \).

Thus by induction, we have \( f(v'_{m(i)}) = k \) for each \( i \leq 2N \).

\( \square \) (Claim 9.2)

It remains only to notice that Claim 9.2 contradicts Lemma 8. So we must have \( v_1 \not\approx v'_1 \).

\( \square \)

For convenience in later statements and arguments, we will use the following notation:
Definition. Given $\sigma \in \Gamma$, define the word $\zeta_N(\sigma)$ by

$$\zeta_N(\sigma) := \begin{cases} 
\alpha_N & \text{if } \sigma = a \\
\beta_N & \text{if } \sigma = b \\
\gamma_N & \text{if } \sigma = c \\
\beta_N d_t & \text{if } \sigma = d_t \text{ (for some } t \in [0, 1]). 
\end{cases}$$

Lemma 10. Suppose $\sigma, \tau \in \Gamma$, $v_0, \ldots, v_\kappa \in V(G)$ are consecutive in $G$, and $v_0', \ldots, v'_\kappa \in V(G)$ are consecutive in $G$, with

$$v_0 \cdots v_\kappa \xrightarrow{w} \zeta_N(\sigma) \quad \text{and} \quad v_0' \cdots v'_\kappa \xrightarrow{w} \zeta_N(\tau).$$

Suppose further that $v_0 \simeq v_0'$ and $v_1 \simeq v_1'$. Then $\sigma \simeq_\Gamma \tau$.

Proof. Suppose for a contradiction that $\sigma \not\simeq_\Gamma \tau$. If $\sigma = a$ and $\tau \in \{b, d_t : t \in [0, 1]\}$, or vice versa, then this contradicts Lemma 9. If one of them is $c$, say $\sigma$, then $w(v_1) = c$ while $w(v_1') = b \not\simeq_\Gamma c$, so this contradicts property (C1) of the chain quasi-order $\leq$. \hfill $\square$

Proposition 11. There is no chain quasi-order for $\rho_N$, for any $N$.

Proof. Suppose for a contradiction that $\leq$ is a chain quasi-order for $\rho_N$. Observe that since $r, p_1, q_1 \in V(G_{\rho_N})$ are all adjacent to $o$ in $G_{\rho_N}$, we have that these three vertices are also adjacent to $o$ in the $\leq$ order. Hence by the pigeonhole principle, some pair of them are $\simeq$. But this is a contradiction by Lemma 10. \hfill $\square$

Oversteegen & Tymchatyn exhibit in [17] for each $\delta > 0$ a 2-dimensional plane strip with span $< \delta$ which has no chain cover of mesh $< 1$. Repovš et al. modify this example in [21] to construct for each $\delta > 0$ a tree in the plane with span $< \delta$ which has no chain cover of mesh $< 1$. In both examples, the diameters of the continua converge to $\infty$ as $\delta \to 0$. We pause to point out that we have now obtained a bounded family of such examples.

Corollary 12. There is a uniformly bounded sequence $\langle T_N \rangle_{N=1}^\infty$ of simple triods in $\mathbb{R}^2$ such that for each $N$, span($T_N$) $< \frac{1}{N}$ and $T_N$ has no chain cover of mesh $< \frac{1}{N}$.

Proof. This is simply a combination of Propositions 1 (using $T_0$ and taking $\epsilon \leq \frac{1}{N}$), 3, 5, and 11. \hfill $\square$

We are working to prove a stronger result: that there is a continuum in $\mathbb{R}^2$ which has span zero and cannot be covered by a chain of mesh less than some positive constant. To this end we will need some further technical combinatorial lemmas.

Lemma 13. Suppose $\sigma, \tau \in \Gamma$ with $\sigma \simeq_\Gamma \tau$, and that $v_0, \ldots, v_\kappa \in V(G)$ are consecutive in $G$ and $v_0', \ldots, v'_\kappa \in V(G)$ are consecutive in $G$ with

$$v_0 \cdots v_\kappa \xrightarrow{w} \zeta_N(\sigma) \quad \text{and} \quad v_0' \cdots v'_\kappa \xrightarrow{w} \zeta_N(\tau).$$

Then:

(i) if $v_0 < v_1$, then $v_0 < v_j < v_\kappa$ for each $0 < j < \kappa$;
(ii) if $v_{k-1} < v_\kappa$, then $v_0 < v_j < v_k$ for each $0 < j < \kappa$;
(iii) if $v_0 \simeq v_0'$ and $v_1 \simeq v_1'$, then $v_\kappa \simeq v'_\kappa$; and
(iv) if $v_\kappa \simeq v'_\kappa$ and $v_{k-1} \simeq v'_{k-1}$, then $v_0 \simeq v_0'$.

Proof. Each of these statements is trivial if $\sigma = \tau = a$. We will prove the Lemma for $\sigma = \tau = a$; the case $\sigma \simeq_\Gamma \tau \simeq_\Gamma b$ proceeds analogously.
(i) Suppose \( v_0 < v_1 \).

**Claim 13.1.** \( v_0 \cdots v_n(0) \xrightarrow{f} (f(v_n(0)) - 6N - 5) \cdots f(v_n(0)). \)

**Proof of Claim 13.1.** This is immediate from Lemma 6. \( \square \)(Claim 13.1)

**Claim 13.2.** For each \( i < 2N \), \( v_{n(i)} \leq v_{n(i+1)} \).

**Proof of Claim 13.2.** We proceed by induction on \( i < 2N \). Suppose the claim is true for each \( i' \) with \( i' < i \). Put \( k := f(v_{n(i)}). \) Suppose for a contradiction that \( f(v_{n(i)}) > f(v_{n(i+1)}). \) By Lemma 7, this means

\[
v_{n(i)} \cdots v_{n(i+1)} \xrightarrow{f} k \cdots (k - 2\theta(i)).
\]

In particular, we have \( f(v_{n(i)+\theta(i)+1}) = k - \theta(i) - 1. \)

Let \( j^* \) be the smallest \( j \leq i \) such that \( f(v_{n(j)}) = k. \)

If \( j^* = 0 \), then since \( n(0) > \theta(i) \), we have \( f(v_{n(0)-\theta(i)-1}) = k - \theta(i) - 1. \)

But also \( w(v_{n(i)+\theta(i)+1}) = c \neq \Gamma a = w(v_{n(i)-\theta(i)-1}) \), so this contradicts property (C1) of the chain quasi-order \( \leq. \)

If \( j^* > 0 \), then we know by Lemma 7 that

\[
v_{n(j^*-1)} \cdots v_{n(j^*+1)} \xrightarrow{f} (k - 2\theta(j^* - 1)) \cdots k.
\]

Then similarly observe that since \( \theta(j^*-1) > \theta(i) \), we have \( f(v_{n(j^*-1)-\theta(i)-1}) = k - \theta(i) - 1. \)

But also \( w(v_{n(i)+\theta(i)+1}) = c \neq \Gamma a = w(v_{n(j^*)-\theta(i)-1}) \), so this contradicts property (C1) of the chain quasi-order \( \leq. \) \( \square \)(Claim 13.2)

**Claim 13.3.** \( v_{n(2N)} \cdots v_\kappa \xrightarrow{f} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) + 6N + 5). \)

**Proof of Claim 13.3.** By Lemma 8 and Claim 13.2, we must have \( v_{n(i-1)} < v_{n(i)} \) for some \( 0 < i \leq 2N \); let \( i^* \) be the largest such \( i \), so that \( f(v_{n(2N)}) = f(v_{n(i^*)}). \)

Suppose for a contradiction that

\[
v_{n(2N)} \cdots v_\kappa \xrightarrow{f} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) - 6N - 5).
\]

Then in particular, since \( 6N + 5 > \theta(i^*-1) \), we have \( f(v_{n(2N)+\theta(i^*-1)+1}) = f(v_{n(2N)}) - \theta(i^*-1) - 1. \)

But also \( f(v_{n(i^*)-\theta(i^*-1)-1}) = f(v_{n(2N)}) - \theta(i^*-1) - 1 \) and \( w(v_{n(i^*)-\theta(i^*-1)-1}) = c \neq \Gamma a = w(v_{n(2N)+\theta(i^*-1)+1}) \), so this contradicts property (C1) of the chain quasi-order \( \leq. \) Therefore by Lemma 6, we must have

\[
v_{n(2N)} \cdots v_\kappa \xrightarrow{f} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) + 6N + 5).
\] \( \square \)(Claim 13.3)

It is now easy to check that \( f(v_0) = f(v_{n(0)}) - 6N - 5 < f(v_j) < f(v_{n(2N)}) + 6N + 5 = f(v_\kappa) \) for any \( 0 < j < \kappa. \)

(ii) Observe that if we consider the reverse order of \( \leq \), part (i) gives that if \( v_0 > v_1 \), then \( v_0 > v_j > v_\kappa \) for each \( 0 < j < \kappa. \) In particular, this would mean \( v_{\kappa-1} > v_\kappa. \) Therefore if \( v_{\kappa-1} < v_\kappa \) then \( v_0 < v_1 \), hence the conclusion follows from part (i).
(iii) Suppose \(v_0 \simeq v'_0\), \(v_1 \simeq v'_1\), and assume without loss of generality that \(v_0 < v_1\). This means Claims 13.1, 13.2, and 13.3 hold for the \(v_j\)'s and the \(v'_j\)'s. By Claim 13.1, we have
\[
v_0 \cdots v_{n(0)} \mapsto (f(v_{n(0)}) - 6N - 5) \cdots f(v_{n(0)})
\]
and
\[
v'_0 \cdots v'_{n(0)} \mapsto (f(v_{n(0)}) - 6N - 5) \cdots f(v_{n(0)}).
\]

Claim 13.4. For each \(i \leq 2N\), \(v_{n(i)} \simeq v'_{n(i)}\).

Proof of Claim 13.4. Suppose not, and let \(i^*\) be the smallest \(i < 2N\) such that \(v_{n(i+1)} \not\simeq v'_{n(i+1)}\). Put \(k := f(v_{n(i^*)}) = f(v'_{n(i^*)})\). It follows from Lemma 7 and Claim 13.2 that either \(f(v_{n(i^*+1)}) = k\) and \(f(v'_{n(i^*+1)}) > k\), or \(f(v_{n(i^*+1)}) > k\) and \(f(v'_{n(i^*+1)}) = k\); assume the former. This implies by Lemma 7 that
\[
v'_{n(i^*)} \cdots v'_{n(i^*+1)} \mapsto k \cdots (k + 2\theta(i^*)).
\]
We claim that \(f(v_{n(i)}) = k\) for each \(i \geq i^*\). Indeed, given \(i > i^*\), suppose for a contradiction that
\[
v_{n(i)} \cdots v_{n(i+1)} \mapsto k \cdots (k + 2\theta(i)).
\]
This means in particular that \(f(v_{n(i)} + \theta(i+1)) = k + \theta(i) + 1\). Since \(\theta(i) < \theta(i^*)\), we have \(f(v'_{n(i^*)}) + \theta(i+1) = k + \theta(i) + 1\). But \(w(v_{n(i)} + \theta(i+1)) = c \not\simeq a = w(v'_{n(i^*)} + \theta(i+1))\), so this contradicts property \((C1)\) of the chain quasi-order \(\leq\). Therefore by Lemma 7 and Claim 13.2, we must have \(f(v_{n(i^*+1)}) = k\). Hence, by induction, \(f(v_{n(i)}) = k\) for each \(i \geq i^*\).

In particular, \(f(v_{n(2N)}) = k\). By Claim 13.3, we have
\[
v_{n(2N)} \cdots v_{n+1} \mapsto k \cdots (k + 6N + 5).
\]
Since \(6N + 5 > \theta(i^*)\), this means that \(f(v_{n(2N)} + \theta(i^*+1)) = k + \theta(i^*) + 1\).
Note \(f(v'_{n(i^*)} + \theta(i^*+1)) = k + \theta(i^*) + 1\) as well. But \(w(v'_{n(i^*)} + \theta(i^*+1)) = c \not\simeq a = w(v_{n(2N)} + \theta(i^*+1))\), so this contradicts property \((C1)\) of the chain quasi-order \(\leq\).

Claim 13.4 implies in particular that \(f(v_{n(2N)}) = f(v'_{n(2N)})\). Then by Claim 13.3, we have
\[
v_{n(2N)} \cdots v_{n+1} \mapsto f(v_{n(2N)}) \cdots (f(v_{n(2N)}) + 6N + 5),
\]
and
\[
v'_{n(2N)} \cdots v'_{n+1} \mapsto f(v'_{n(2N)}) \cdots (f(v'_{n(2N)}) + 6N + 5).
\]
This establishes part (iii).

(iv) Suppose \(v_n \simeq v'_n\), \(v_{n-1} \simeq v'_{n-1}\), and assume without loss of generality that \(v_{n-1} < v_n\). By part (ii) this implies \(v_0 < v_1\) and \(v'_0 < v'_1\), so again Claims 13.1, 13.2, and 13.3 hold for the \(v_j\)'s and the \(v'_j\)'s. By Claim 13.3, we have
\[
v_n \cdots v_{n(2N)} \mapsto (f(v_{n(2N)}) + 6N + 5) \cdots f(v_{n(2N)})
\]
and
\[
v'_n \cdots v'_{n(2N)} \mapsto (f(v'_{n(2N)}) + 6N + 5) \cdots f(v'_{n(2N)}).
\]
Claim 13.5. For each $i \leq 2N$, $v_{n(i)} \simeq v'_{n(i)}$.

Proof of Claim 13.5. Suppose not, and let $i^*$ be the largest $i < 2N$ such that $v_{n(i)} \not\simeq v'_{n(i)}$. Put $k := f(v_{n(i^*+1)}) = f(v'_{n(i^*+1)})$. It follows from Lemma 7 and Claim 13.2 that either $f(v_{n(i^*)}) = k$ and $f(v'_{n(i^*)}) < k$, or $f(v_{n(i^*)}) < k$ and $f(v'_{n(i^*)}) = k$; assume the former. This implies by Lemma 7 that

$$v'_{n(i^*+1)} \cdots v'_{n(i^*)} \nearrow k \cdots (k - 2\theta(i^*))$$

We claim that $f(v_{n(i)}) = k$ for each $i \leq i^*$. Indeed, given $i < i^*$, suppose for a contradiction that

$$v_{n(i+1)} \cdots v_{n(i)} \nearrow k \cdots (k - 2\theta(i)).$$

Since $\theta(i^*) < \theta(i)$, this means in particular that $f(v_{n(i+1)} - \theta(i^*) - 1) = k - \theta(i^*) - 1$. Note $f(v_{n(i^*+1)} - \theta(i^*) - 1) = k - \theta(i^*) - 1$ as well. But $w(v_{n(i^*+1)} - \theta(i^*) - 1) = c \not\simeq \Gamma a = w(v'_{n(i^*+1)} - \theta(i^*) - 1)$, so this contradicts property (C1) of the chain quasi-order $\leq$. Therefore by Lemma 7 and Claim 13.2 we must have $f(v_{n(i)}) = k$. Hence, by induction, $f(v_{n(i)}) = k$ for each $i \leq i^*$.

In particular, $f(v_{n(0)}) = k$. By Claim 13.1, we have

$$v_{n(0)} \cdots v_{0} \nearrow k \cdots (k - 6N - 5).$$

Since $6N + 5 > \theta(i^*)$, this means that $f(v_{n(0)} - \theta(i^*) - 1) = k - \theta(i^*) - 1$. Note $f(v'_{n(i^*+1)} - \theta(i^*) - 1) = k - \theta(i^*) - 1$ as well. But $w(v'_{n(i^*+1)} - \theta(i^*) - 1) = c \not\simeq \Gamma a = w(v_{n(0)} - \theta(i^*) - 1)$, so this contradicts property (C1) of the chain quasi-order $\leq$. \hfill \Box(\text{Claim 13.5})

Claim 13.5 implies in particular that $f(v_{n(0)}) = f(v'_{n(0)})$. Then by Claim 13.1, we have

$$v_{n(0)} \cdots v_{0} \nearrow f(v_{n(0)}) \cdots (f(v_{n(0)}) - 6N - 5)$$

and

$$v'_{n(0)} \cdots v'_{0} \nearrow f(v_{n(0)}) \cdots (f(v'_{n(0)}) - 6N - 5).$$

This establishes part (iv).

\hfill \Box

4.2. Iterated sketches. If $\nu_T : \Gamma \to T$ is a $\Gamma$-marking of the simple triod $T$ and $\rho_N$ is a $(T, \varepsilon)$-sketch of the simple triod graph $T' := G_{\rho_N}$ such that $|\bar{q}|_{\beta_N[2]} = |\bar{q}_{\beta_N[2]}| = [\nu_T(c), \nu_T(b)]$ (as in Proposition 1), then one can define an induced $\Gamma$-marking $\nu_{T'} : \Gamma \to T'$ on $T'$ as follows: define $\nu_T(a) := p_{|\alpha_N| - 1}$, $\nu_T(b) := q_{|\beta_N[1]| - 1} = \nu_T(b)$, $\nu_T(c) := r$, and for each $t \in [0, 1]$ put $\nu_{T'}(dt) := \nu_T(dt) \in [\bar{q}_{|\beta_N[2]} - 2, \bar{q}_{|\beta_N[2]} - 1] = [\nu_T(c), \nu_T(b)]$.

Now let $T_0$ be as before, and suppose $T_1$ and $T_2$ are simple triods such that $\rho_1$ is a $(T_0, \varepsilon_0)$-sketch of $T_1$, and $\rho_2$ is a $(T_1, \varepsilon_1)$-sketch of $T_2$ (using the induced $\Gamma$-marking on $T_1$). Evidently we should be able to find a $(T_0, \varepsilon_0 + \varepsilon_1)$-sketch of $T_2$, and indeed this is necessary if we want to apply Proposition 5 to argue that $T_2$ has no chain cover of small mesh. This is the motivation for the next definition (see Proposition 14).
Proposition 14. Suppose \( \langle G, w \rangle \) is a compliant graph-word, and \( N > 0 \). A graph-word \( \langle G^+, w^+ \rangle \) is a \( \rho_N \)-expansion of \( \langle G, w \rangle \) if:

- \( G^+ \) is identical to \( G \) as a topological space, but the vertex set of \( G^+ \) is finer: for any adjacent pair of vertices \( v_1, v_2 \in V(G) \), there are distinct degree 2 vertices \( s_{j_1}^{v_1 v_2}, j = 1, \ldots, \kappa_{v_1 v_2} \) where \( \kappa_{v_1 v_2} = [\zeta_N(w(v_1))] + [\zeta_N(w(v_2))] - 3 \), inserted into the edge joining \( v_1, v_2 \) so that \( v_1, s_{j_1}^{v_1 v_2}, \ldots, s_{\kappa_{v_1 v_2}}^{v_1 v_2}, v_2 \) are consecutive in \( G^+ \); and

- \( w^+ \) is defined by

\[
v_1 s_1^{v_1 v_2} \cdots s_{\kappa_{v_1 v_2}}^{v_1 v_2} v_2 \xrightarrow{w^+} \zeta_N(w(v_1)) \quad \text{in} \quad \zeta_N(w(v_2))
\]

when \( v_1, v_2 \in V(G) \) are adjacent in \( G \).

Remarks. 1) Notice that \( w^+|_{V(G)} = w \), and that \( \langle G^+, w^+ \rangle \) is also a compliant graph-word.

2) Combinatorially, there is only one \( \rho_N \)-expansion of a given graph-word \( \langle G, w \rangle \); however, geometrically they may differ according to where along the edges of \( G \) the extra vertices are inserted (though their order on the edge is determined uniquely by the definition).

Proposition 14. Suppose \( T \) is a \( \Gamma \)-marked simple triod, and \( \rho_N \) is a \( \langle T, \varepsilon_1 \rangle \)-sketch of \( T' := G_{N^+} \). Endow \( T' \) with a \( \Gamma \)-marking as above. If \( \rho = \langle G, w \rangle \) is a compliant graph-word which is a \( \langle T', \varepsilon_2 \rangle \)-sketch of \( G \), then there is a \( \rho_N \)-expansion of \( \langle G, w \rangle \) which is a \( \langle T, \varepsilon_1 + \varepsilon_2 \rangle \)-sketch of \( G \).

Proof. Let \( \hat{w}_{\rho_N} : T' \to T \) be a \( \rho_N \)-suggested bonding map such that \( d(x, \hat{w}_{\rho_N}(x)) < \frac{\varepsilon_2}{2} \) for each \( x \in T' \), and let \( \hat{w} : G \to T' \) be \( \rho \)-suggested bonding map such that \( d(x, \hat{w}(x)) < \frac{\varepsilon_2}{2} \) for each \( x \in G \).

Consider any adjacent \( v_1, v_2 \in V(G) \). Define

\[
s_{j_1}^{v_1 v_2} := \begin{cases} \hat{w}^{-1}(p_{|\alpha_N|+1-i}) & \text{if } w(v_1) = a \\ \hat{w}^{-1}(q_{|\beta_N|+1-i}) & \text{if } w(v_1) \approx_G b \end{cases}
\]

for \( 1 \leq i \leq |\zeta_N(w(v_1))| \), and

\[
s_{\kappa_{v_1 v_2}}^{v_1 v_2} := \begin{cases} \hat{w}^{-1}(p_{|\alpha_N|+1-i}) & \text{if } w(v_2) = a \\ \hat{w}^{-1}(q_{|\beta_N|+1-i}) & \text{if } w(v_2) \approx_G b \end{cases}
\]

for \( 1 \leq i \leq |\zeta_N(w(v_2))| \).

Let \( V(G^+) \) be equal to \( V(G) \) together with all these new vertices, and let \( w^+ \) be defined as in the definition of a \( \rho_N \)-expansion. Observe that \( w^+ = \rho_N \circ \hat{w}|_{V(G^+)} \).

Put \( \rho^+ := \langle G^+, w^+ \rangle \), where \( G^+ \) is equal to \( G \) as a topological space, with vertex set \( V(G^+) \).

It is now straightforward to see that \( \hat{w}_{\rho_N} \circ \hat{w} \) is a \( \rho^+ \)-suggested bonding map, and clearly \( d(x, (\hat{w}_{\rho_N} \circ \hat{w})(x)) < \frac{\varepsilon_1 + \varepsilon_2}{2} \) for each \( x \in G \).

\( \square \)

Lemma 15. Suppose \( \langle G, w \rangle \) is a compliant graph-word, let \( \langle G^+, w^+ \rangle \) be a \( \rho_N \)-expansion of \( \langle G, w \rangle \), and suppose \( \leq^+ \) is a chain quasi-order of \( \langle G^+, w^+ \rangle \).

(i) Let \( v_1, v_2 \in V(G) \) be adjacent in \( G \), and let \( s_1, \ldots, s_\kappa \in V(G^+) \) together with \( V(G) \) be such that \( v_1, s_1, \ldots, s_\kappa, v_2 \) are consecutive in \( G^+ \). Then the following are equivalent:

1) \( v_1 \prec^+ v_2 \);
(2) $v_1 \prec^+ s_j \prec^+ v_2$ for each $j \in \{1, \ldots, \kappa\};$
(3) $v_1 \prec^+ s_j \prec^+ v_2$ for some $j \in \{1, \ldots, \kappa\}.$

(ii) If $v_1, v_2 \in V(G)$ are adjacent in $G$ and $v'_1, v'_2 \in V(G)$ are adjacent in $G$
with $v_1 \succeq^+ v'_1$, $v_1 \prec^+ v'_2$, and $v'_1 \prec^+ v'_2$, then $v_2 \succeq^+ v'_2.$

Proof. (i) The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are trivial. For (1) $\Rightarrow$
(2) we will prove that $v_1 \prec^+ s_j$ implies that $v_1 \prec^+ s_j \prec^+ v_2$ for each $j \in \{1, \ldots, \kappa\}$. Then by considering the reverse order of $\leq^+$, it follows that $v_1 \prec^+ v_2$ implies $v_1 \prec^+ s_j$, hence $v_1 \prec^+ s_j \prec^+ v_2$ for each $j \in \{1, \ldots, \kappa\}$.

Suppose $v_1 \prec^+ s_1$. Let $i \in \{1, \ldots, \kappa\}$ be such that $s_i \cdots s_1 v_1 \xrightarrow{w^+} \zeta_N(w(v_1))$ and $s_i \cdots s_\kappa v_2 \xrightarrow{w^+} \zeta_N(w(v_2)).$

By Lemma 13 (ii), we have $v_1 \prec^+ s_j \prec^+ s_i$ for each $j \in \{1, \ldots, i - 1\}$. Because $G$ is compliant, we can deduce using Lemma 10 that $s_i \prec^+ s_{i+1}$. Then by Lemma 13 (i) we have $s_i \prec^+ s_j \prec^+ v_2$ for each $j \in \{i + 1, \ldots, \kappa\}$.

(ii) Suppose $v_1, v_2 \in V(G)$ are adjacent in $G$ and $v'_1, v'_2 \in V(G)$ are adjacent
in $G$ with $v_1 \succeq^+ v'_1$, $v_1 \prec^+ v'_2$, and $v'_1 \prec^+ v'_2$. Let $s_1, \ldots, s_\kappa$ and $i$ be as in
part (i), and let $s_1', s_1, \ldots, s_\kappa'$, $s'_1 \cdots s_\kappa' v_2 \in V(G^+) \setminus V(G)$ be such that $v'_1 s'_1, \ldots, s'_\kappa v'_2$
are consecutive in $G^+$ and $v'_1 s'_1 \cdots s'_\kappa v'_2 \xrightarrow{w^+} \zeta_N(w(v'_1)) \setminus \zeta_N(w(v'_2)).$

By property (C1) of the chain quasi-order $\leq^+$, $w(v_1) \approx_{V} w(v'_1)$, hence $|\zeta_N(v_1)| = |\zeta_N(v'_1)|$, and so $s'_1 \cdots s'_i v'_1 \xrightarrow{w^+} \zeta_N(w(v'_1))$ and $s'_1 \cdots s'_\kappa v'_2 \xrightarrow{w^+} \zeta_N(w(v'_2))$.

By Lemma 13 (iv), we have $s_1 \prec^+ s_i'$, and as in part (i) we know that $s_i' \prec^+ s'_1$. By Lemma 10, this implies $w(v_2) \approx_{V} w(v'_2)$, hence $\kappa = \lambda$. Then by Lemma 13 (iii), we conclude that $v_2 \succeq^+ v'_2$. 

\[ \Box \]

**Proposition 16.** Suppose $\langle G, w \rangle$ is a compliant graph-word. If a (any) $\rho_N$-expansion
of $\langle G, w \rangle$ has a chain quasi-order, then $\langle G, w \rangle$ also has a chain quasi-order.

Proof. Let $\langle G^+, w^+ \rangle$ be a $\rho_N$-expansion of $\langle G, w \rangle$, and let $\leq^+$ be a chain quasi-order
of $G^+, w^+$. Define $\leq$ on $V(G)$ by $\leq := \leq^+|V(G)$. Clearly $\leq$ is a total quasi-order since $\leq^+$ is. We must check that $\leq$ satisfies properties (C1), (C2), and (C3) of the definition
of a chain quasi-order.

(C1): This is immediate since $\leq^+$ satisfies this property.

(C2): We will need the following claim:

**Claim 16.1.** In $\langle G^+, w^+ \rangle$, if $v \in V(G)$ and $v' \in V(G^+)$ are such that $v \succeq^+ v'$,
then in fact $v' \in V(G)$.

**Proof of Claim 16.1.** We proceed by induction on the number of vertices in $G$.

If $|V(G)| = 1$, then there is nothing to prove.

Assume the claim holds for all such graph-words whose graph has $n$ or fewer vertices, and assume $|V(G)| = n + 1$. Let $u \in V(G)$ be such that the subgraph $G^-$
obtained by removing the vertex $u$ (and all edges emanating from $u$) is connected. There is a $\rho_N$-expansion of $\langle G^-, w|_{V(G) \setminus \{u\}} \rangle$ which is a sub-graph-word of $\langle G^+, w^+ \rangle$.
(it has vertex set $V(G^+ \cap G^-)$, and the restriction of $\leq^+$ to this sub-graph-word is a chain quasi-order. By induction, the claim holds for $G^-$. Let $u' \in V(G) \setminus \{u\}$ be adjacent to $u$ in $G$. Let $s_1, \ldots, s_\kappa \in V(G^+) \setminus V(G)$ be such that $u', s_1, \ldots, s_\kappa, u$ are consecutive in $G^+$ and

$$u's_1 \cdots s_\kappa u \xrightarrow{w^+} \zeta_N(w(u')) \sqcap \zeta_N(w(u)).$$

Assume $u' <^+ u$ (the other case proceeds similarly), which implies by Lemma 15 (i) that $u' <^+ s_j <^+ u$ for each $j \in \{1, \ldots, \kappa\}$.

We have four things to check:

1. For each $y \in V(G) \setminus \{u\}$ and each $s \in V(G^+) \setminus V(G)$ in the $\rho_N$-expansion of $G^-$, $y \not\sim^+ s$;
2. For each $y \in V(G) \setminus \{u\}$ and each $j \in \{1, \ldots, \kappa\}$, $y \not\sim^+ s_j$;
3. For each $s \in V(G^+) \setminus V(G)$ in the $\rho_N$-expansion of $G^-$, $u \not\sim^+ s$; and
4. For each $j \in \{1, \ldots, \kappa\}$, $u \not\sim^+ s_j$.

Observe that (1) holds by induction, and (4) is immediate from the fact that $u' <^+ s_j <^+ u$ for each $j \in \{1, \ldots, \kappa\}$. For (2) and (3), we consider two cases.

**Case 1.** For every $y \in V(G) \setminus \{u\}$, $y \leq^+ u'$.

Since $u' <^+ s_j <^+ u$ for each $j \in \{1, \ldots, \kappa\}$, we have immediately that $y \not\sim^+ s_j$ for any $y \in V(G) \setminus \{u\}$.

Also, from Lemma 15 (i) it follows that for every $s \in V(G^+) \setminus V(G)$ in the $\rho_N$-expansion of $G^-$, $s <^+ u'$. Therefore $u \not\sim^+ s$ for any such $s$.

**Case 2.** There exists some $y \in V(G) \setminus \{u\}$ such that $u' <^+ y$.

Let $P$ be a path of vertices in $G^-$ starting at $u'$ and ending at $y$. Let $y_1$ be the latest vertex $y'$ in $P$ with $y' \leq^+ u'$, and let $y_2$ be the next vertex in $P$ after $y_1$, so that $y_1$ and $y_2$ are adjacent in $G$ and $y_1 \leq^+ u' <^+ y_2$.

Suppose for a contradiction that $y_1 <^+ u'$. Let $z_1, \ldots, z_\lambda \in V(G^+) \setminus V(G)$ be such that $y_1, z_1, \ldots, z_\lambda, y_2$ are consecutive in $G^+$, then by Lemma 4 there is some $i \in \{1, \ldots, \lambda\}$ such that $u' \sim^+ z_i$. But this contradicts the fact that the claim holds for $G^-$ by induction. Therefore we must have $u' \sim^+ y_1$.

Then from Lemma 15 (ii) we know that $u \sim^+ y_2$. It follows immediately that $u \not\sim^+ s$ for each $s \in V(G^+) \setminus V(G)$ in the $\rho_N$-expansion of $G^-$, because $y_2 \not\sim^+ s$ for every such $s$ by induction.

Moreover, for each $j \in \{1, \ldots, \kappa\}$, since $y_1 \sim^+ u' <^+ s_j <^+ u \sim^+ y_2$, we know from Lemma 4 that there is some $s \in V(G^+) \setminus V(G)$ inserted between $y_1$ and $y_2$ such that $s_j \sim^+ s$. It follows that $y \not\sim^+ s_j$ for any $y \in V(G) \setminus \{u\}$, because $y \not\sim^+ s$ for every such $y$ by induction.

\[\square\text{(Claim 16.1)}\]

Now suppose $v_1, v_2 \in V(G)$ are adjacent in $G$, and assume $v_1 < v_2$. Let $s_1, \ldots, s_\kappa \in V(G^+) \setminus V(G)$ be such that $v_1, s_1, \ldots, s_\kappa, v_2$ are consecutive in $V(G^+)$. If $v \in V(G)$ were such that $v_1 < v < v_2$, then $v_1 <^+ v <^+ v_2$ as well, so by Lemma 4 there would be some $i \in \{1, \ldots, \kappa\}$ such that $v \sim^+ s_i$. But this contradicts Claim 16.1.

(C3): Suppose $v \in V(G), v_1, v_2, v_3$ are consecutive in $G$, and that $\sigma, \tau \in \{a, c\}$ and $t, t' \in [0, 1]$ are such that $t' \geq t, w(v) = d_{v'}, v_1 v_2 v_3 \xrightarrow{w^+} \sigma d_t \tau$, and $v_1 < v_2 \sim v < v_3.$
Let \( s_1, \ldots, s_\kappa, s'_1, \ldots, s'_\lambda \in V(G^+) \setminus V(G) \) be such that \( v_1, s_1, \ldots, s_\kappa, v_2, s'_1, \ldots, s'_\lambda, v_3 \) are consecutive in \( G^+ \), and

\[
v_1 s_1 \cdots s_\kappa v_2 s'_1 \cdots s'_\lambda v_3^{w^+} \xrightarrow{\nu} \zeta_N(\sigma)^{+} \ni \beta_N^{-}(\beta_N^{-})^{+} \ni \zeta_N(\tau).
\]

Observe that \( w^+ (s_\kappa) = w^+ (s'_\lambda) = c \). Since \( v_1 < v_2 \), by Lemma 15 (i) we must have \( s_\kappa < v_2 \). Likewise, we have \( v_2 < v_3^+ \). It now follows from property (C3) of the chain quasi-order \( \leq^+ \) that \( t' - t < \frac{1}{2} \).

\( \square \)

5. The Example

**Example.** There exists a continuum \( X \subset \mathbb{R}^2 \) which is non-chainable and has span zero.

**Proof.** First we define by recursion a sequence \( \langle T_N \rangle_{N=0}^\infty \) of simple triods in \( \mathbb{R}^2 \) and a sequence \( \langle \varepsilon_N \rangle_{N=0}^\infty \) of positive reals as follows.

Let \( T_0 \subset \mathbb{R}^2 \) be as defined above, and put \( \varepsilon_0 := \frac{1}{8} \).

Suppose \( T_N, \varepsilon_N \) have been defined. Apply Proposition 1 to obtain an embedding \( T_{N+1} \) of the simple triod graph \( G_{\rho_{N+1}} \) in \( \mathbb{R}^2 \) such that \( \rho_{N+1} \) is a \( \langle T_N, \varepsilon_N \rangle \)-sketch of \( T_{N+1} \). Endow \( T_{N+1} \) with a \( \Gamma \)-marking as above. Notice that \( T_{N+1} \subset \langle T_N \rangle_{\varepsilon_N} \), where \( Y_\varepsilon \) denotes the \( \varepsilon \)-neighborhood of the space \( Y \). By Proposition 3, the span of \( T_{N+1} \) is \( < \frac{1}{2(2N+1)} + \varepsilon_N \). Let \( 0 < \varepsilon_{N+1} < 2^{-N-4} \) be small enough so that

\[
\langle T_{N+1} \rangle_{\varepsilon_{N+1}} \subseteq \langle T_N \rangle_{\varepsilon_N}, \text{ and so that span}(\langle T_{N+1} \rangle_{\varepsilon_{N+1}}) < \frac{1}{2(2N+1)} + 2 \varepsilon_N.
\]

Put \( X := \bigcap_{N=0}^\infty \langle T_N \rangle_{\varepsilon_N} \).

Observe that for any \( N \), we have \( X \subseteq \langle T_{N+1} \rangle_{\varepsilon_{N+1}} \), hence

\[
\text{span}(X) \leq \text{span}(\langle T_{N+1} \rangle_{\varepsilon_{N+1}}) < \frac{1}{2(2N+1)} + 2 \varepsilon_N.
\]

Since \( \varepsilon_N \) converges to 0 as \( N \to \infty \), it follows that \( X \) has span zero.

Suppose for a contradiction that \( X \) has a chain cover of mesh \( < \frac{1}{4} \). Then there is some \( N > 0 \) for which \( T_N \) has a chain cover of mesh \( < \frac{1}{4} \).

Define by recursion the graph-words \( \langle G_i, w_i \rangle \), \( 0 \leq i \leq N - 1 \), as follows: \( \langle G_{N-1}, w_{N-1} \rangle := \rho_N \), and for \( i < N - 1 \), \( \langle G_i, w_i \rangle \) is the \( \rho_{i+1} \)-expansion of \( \langle G_{i+1}, w_{i+1} \rangle \) provided by Proposition 14 which is a \( \langle T_i, \sum_{j=i}^{N-1} \varepsilon_j \rangle \)-sketch of \( T_N \). In particular, \( \langle G_0, w_0 \rangle \) is a \( \langle T_0, \sum_{j=0}^{N-1} \varepsilon_j \rangle \)-sketch of \( T_N \).

Since \( \sum_{j=0}^{N-1} \varepsilon_j < \sum_{j=0}^{N-1} 2^{-j-3} < \frac{1}{4} \), by Proposition 5 we have that \( \langle G_0, w_0 \rangle \) has a chain quasi-order. Then by Proposition 16 and induction, we obtain a chain quasi-order for each graph-word \( \langle G_i, w_i \rangle \). In particular, \( \langle G_{N-1}, w_{N-1} \rangle \) has a chain quasi-order. But \( \langle G_{N-1}, w_{N-1} \rangle = \rho_N \), so this contradicts Proposition 11.

\( \square \)

6. Questions

The construction presented in this paper can be carried out so that every proper subcontinuum of the resulting space is an arc; hence, in particular, it is far from being hereditarily indecomposable. On the other hand, it follows from results of [17] that if there exists a non-degenerate homogeneous continuum in the plane which is not homeomorphic to the circle, the pseudo-arc, or the circle of pseudo-arcs, then there would be one which is hereditarily indecomposable and with span zero. Given that the pseudo-arc is the only hereditarily indecomposable chainable continuum, this raises the following question:
Question 1 (See Problem 9 of [18]). Is there a hereditarily indecomposable non-chainable continuum with span zero?

If such an example exists, then by [19, Corollary 6] it would be a continuous image of the pseudo-arc. Since any map to a hereditarily indecomposable continuum is confluent [22, Lemma 15], it would also be a counterexample to Problem 84 of [4], which asks whether every confluent image of a chainable continuum is chainable.

Regarding the planarity of the example in this paper, while every chainable continuum can be embedded in the plane [2], the same is not known to be true of continua with span zero.

Question 2. Can every continuum with span zero be embedded in $\mathbb{R}^2$?

References

1. Dana Bartošová, Klaas Pieter Hart, L. C. Hoehn, and Berd Van der Steeg, Lelek’s problem is not a metric problem, preprint.