Chapter 3
Boundary Value Problems, Introduction

3.1 The method of images There are some problems in electrostatics for which a solution can be obtained by adding 'image' charges outside the region of interest. The most obvious one is the potential of a charge above a conducting, infinite plane. This is solved by noting that the plane which is the perpendicular bisector of the line between charges of opposite sign but the same magnitude is an equipotential. In fact if the potential at infinity is zero, the potential of the plane will equal the potential at an infinite distance form the charge.

3.1.1 The potential due to a charge above an infinite plane Let the plane in this system lie at \( z = 0 \) and let the charge \( q \) lie at \((x_0, y_0, z_0)\). The potential in the half space \( z > 0 \) satisfies

\[
\nabla^2 \phi (r) = -4\pi k_1 \rho (r') = -4\pi k_1 q \delta^{(3)} (r - r_0)
\]

\( (3.1) \)

\[
= -4\pi q \delta^{(3)} (r - r_0) \quad k_1 = 1 \text{ in Gaussian units}
\]

\[
= -\frac{q}{\varepsilon_0} \delta^{(3)} (r - r_0) \quad k_1 = \frac{1}{4\pi \varepsilon_0} \text{ in SI units}
\]

The general solution to Poisson’s equation in the upper half plane is

\[
\phi (r) = k_1 \iiint \frac{\rho (r')}{|r - r'|} d^2 r' + E_0 \cdot r + V_0 + V (r)
\]

\( (3.2) \)

\[
\phi_1 (r) = \frac{qk_1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}}
\]

\[
- \frac{qk_1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2}}
\]

\( (3.3) \)

Here the second term is \( V (r) \) for a point charge \(-q\) at \( z = -z_0\). \( \phi_1 (r) \) is the potential of a ‘dipole’ with charge separation \( 2z_0\).

The requirement that the normal component of the electric field vanish at \( z = 0 \) provides another special solution for the problem with charge \( q \) in the upper half plane. In this case \( \phi (r) = \phi_2 (r): \)

\[
\phi_2 (r) = \frac{qk_1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}}
\]

\[
+ \frac{qk_1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2}}
\]

\( (3.4) \)
\[
\lim_{z \to 0^+} [\hat{z} \cdot - \nabla \phi_2(r)] = \frac{qk_1}{[(x-x_0)^2 + (y-y_0)^2 + z_0^2]^{3/2}} \lim_{z \to 0^+} [(z-z_0) + (z+z_0)] = 0 \quad (3.5)
\]

Note that the normal component of the electric field, \( E_1 = -\nabla \phi_1(r) \) is not zero at \( z = 0 \).

\[
\lim_{z \to 0^+} [\hat{z} \cdot - \nabla \phi_1(r)] = \frac{qk_1}{[(x-x_0)^2 + (y-y_0)^2 + z_0^2]^{3/2}} \lim_{z \to 0^+} [(z-z_0) - (z+z_0)] \quad (3.6)
\]

But the tangential component of \( E_1 = -\nabla \phi_1(r) \) vanishes at \( z = 0 \).

\[
\lim_{z \to 0^+} [\hat{z} \times - \nabla \phi_1(r)] = \frac{qk_1}{[(x-x_0)^2 + (y-y_0)^2 + z_0^2]^{3/2}} \lim_{z \to 0^+} \hat{z} \times [(r - r_0^+) - (r - r_0^-)] \quad (3.7)
\]

### 3.1.2 Attraction between a charged particle and a conducting plane

The tangential component of the electric field at the surface of a conductor is zero. It follows that \( \phi_1(r) \) provides a solution for the potential of the isolated charge - plane system. The electric field in the \( z > 0 \) half-plane is the sum of the electric field of the point charge and the electric field of the surface charge on the plane. The surface charge on the plane satisfies \( \sigma_1(x, y) = \epsilon_0 \frac{-\partial \phi_1(r)}{\partial z} \) (in S.I. units) evaluated at \( z = 0 \). Thus from Eq. 3.6,

\[
\sigma_1(x, y) = \frac{1}{2\pi} \frac{-qz_0}{[(x-x_0)^2 + (y-y_0)^2 + z_0^2]^{3/2}}. \quad \epsilon_e = \frac{\sigma}{\epsilon_0} \quad (3.8)
\]

Without loss of generality but with some savings in writing we can take the charge to lie at \( r_0 = (0, 0, z_0) \). Because of symmetry the force will not have an \( x \) or \( y \) component so we only need to calculate the \( z \) component. With these simplifying observations the force, \( F_z = q\hat{z} \cdot E_1(r_0) \), on the charge, \( q \), due to the surface charge is

\[
F_z = \frac{q}{4\pi \epsilon_0} \iiint \frac{\rho_1(r') (z_0 - z') d^3x'}{[x'^2 + y'^2 + z_0^2]^{3/2}} \text{ in S.I units} \quad (3.9)
\]

where \( \rho_1(r') = \sigma_1(x', y') \delta(z') \nabla z' \). The result is

\[
F_z = -\frac{q^2 z_0}{2\pi 4\pi \epsilon_0} \int \int \frac{z_0}{[x'^2 + y'^2 + z_0^2]^{3/2}} dx' dy' = -\frac{q^2 z_0^2}{4\pi \epsilon_0} \int \frac{1}{\rho'^2} \frac{d\rho'^2}{[\rho'^2 + z_0^2]^{3/2}} \quad (3.10)
\]

The integration yields \( F_z = -q^2 k_1/4z_0^2 \), just the force between the charge and the ‘image’ charge.
The potential due to a charge inside a spherical volume

Again Eqs. 3.1 and 3.2 provide the general solution for the potential inside a spherical volume, radius \( R \), with a charge \( q \) located at \( \mathbf{r}_0 = (x_0, y_0, z_0) \).

A special solution exists for which the potential vanishes on the sphere. This solution can be obtained by placing an image charge outside the spherical region. We will let the charge be \( q' (R, r_0) \) located at \( \alpha (R, r_0) \) where we have noted the possible dependence of the parameters \( q' \) and \( \alpha \). In this case the potential for \( r < R \) is

\[
\phi_3 (\mathbf{r}) = k_1 \left[ \frac{q}{|\mathbf{r} - \mathbf{r}_0|} + \frac{q' (R, r_0)}{|\mathbf{r} - \alpha (R, r_0) \mathbf{r}_0|} \right].
\]

(3.11)

The requirement that \( \phi_3 (\mathbf{r}) = 0 \) for \( r = R \) is

\[
\frac{q}{|R\hat{\mathbf{r}} - \mathbf{r}_0|} = -\frac{q' (R, r_0)}{|R\hat{\mathbf{r}} - \alpha (R, r_0) \mathbf{r}_0|}
\]

or

\[
\frac{Rq}{r_0 |R\hat{\mathbf{r}} - \mathbf{r}_0|} = -\frac{q' (R, r_0)}{|\hat{\mathbf{r}} - \alpha (R, r_0) \frac{r_0}{R}\hat{\mathbf{r}}_0|}
\]

which is satisfied when

\[
q' (R, r_0) = -q \frac{R}{r_0} \quad \text{and} \quad \alpha (R, r_0) = \frac{R^2}{r_0^2}
\]

(3.12)

so that

\[
\phi_3 (\mathbf{r}) = k_1 \left[ \frac{q}{|\mathbf{r} - \mathbf{r}_0|} - \frac{qR/r_0}{|\mathbf{r} - \frac{R^2}{r_0^2} \mathbf{r}_0|} \right]
\]

(3.12b)

Note that \( \left| \frac{R}{r_0} \hat{\mathbf{r}} - \hat{\mathbf{r}}_0 \right| = \left| \hat{\mathbf{r}} - \frac{R}{r_0} \hat{\mathbf{r}}_0 \right| = [1 + \left( \frac{R}{r_0} \right)^2 - 2 \frac{R}{r_0} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}_0]^{1/2} \). One can also write

\[
|R\hat{\mathbf{r}} - r_0 \hat{\mathbf{r}}_0| = |r_0 \hat{\mathbf{r}} - R\hat{\mathbf{r}}_0|
\]

The total surface charge on the bounding sphere will be \( q \). (The integral of the normal component of the ‘image’ charge electric field over the sphere will vanish.)
\( \phi_3(\mathbf{r}) \) is the potential for a spherical volume which vanishes on the spherical surface bounding the volume. This allows us to identify the Dirichlet Green’s function for the sphere,

\[
G_D(\mathbf{r}, \mathbf{r}') = \frac{-1}{4\pi} \left[ \frac{1}{|\mathbf{r} - \mathbf{r}'|} + \frac{-R}{r' - (R^2/r^2)\mathbf{r}' \cdot \mathbf{r}} \right].
\]

A check shows that \( G_D(\mathbf{r}, \mathbf{r}') = G_D(\mathbf{r}', \mathbf{r}) \):

\[
G_D(\mathbf{r}, \mathbf{r}') = \frac{-1}{4\pi} \left[ \frac{1}{|\mathbf{r} - \mathbf{r}'|} + \frac{-R/(rr')}{\mathbf{r}' \cdot \mathbf{r}} \right] = \frac{-1}{4\pi} \left[ \frac{1}{|\mathbf{r} - \mathbf{r}'|} + \frac{-R/(rr')}{|\mathbf{r}' \cdot \mathbf{r}|} \right] = \frac{-1}{4\pi} \left[ \frac{1}{|\mathbf{r} - \mathbf{r}'|} + \frac{-R/|\mathbf{r}' \cdot \mathbf{r}|}{|\mathbf{r}' \cdot \mathbf{r}|} \right].
\]

Thus,

\[
G_D(\mathbf{r}, \mathbf{r}') = \frac{-1}{4\pi} \left[ \frac{1}{|\mathbf{r} - \mathbf{r}'|} + \frac{-R/|\mathbf{r}' \cdot \mathbf{r}|}{|\mathbf{r}' \cdot \mathbf{r}|} \right] = \frac{-1}{4\pi} \left[ \frac{1}{|\mathbf{r} - \mathbf{r}'|} + \frac{-R/|\mathbf{r}' \cdot \mathbf{r}|}{|\mathbf{r} - \mathbf{r}'|} \right].
\]

Note that \( a = \frac{R^2}{r} \) is along the \( \mathbf{r} \) direction and \( \frac{a}{R} = \frac{R}{r} \). Thus whenever \( r \) is inside the sphere \( (r < R) \), \( a \) is outside the sphere \( (R < a) \) and whenever \( r \) is outside the sphere \( (r > R) \), \( a \) is inside the sphere \( (R > a) \). It is convenient to think of \( a \) as the position of the image charge. We can also write:

\[
G_D(\mathbf{r}, \mathbf{r}') = \frac{-1}{4\pi} \left[ \frac{1}{|\mathbf{r} - \mathbf{r}'|} + \frac{-R/|\mathbf{r}' \cdot \mathbf{r}|}{|\mathbf{r} - \mathbf{r}'|} \right] ; \quad a = \frac{R^2}{r} \quad \frac{a}{R} = \frac{R}{r} \quad (3.14a)
\]

We note also that we can express the potential at the point \( \mathbf{r} \) (inside the sphere of radius \( R \)) due to a point charge \( q \) at \( \mathbf{r}' \) inside the sphere, with the condition that \( \Phi(R\mathbf{r}) = 0 \).

\[
\Phi(\mathbf{r}) = \frac{q}{4\pi} \left[ \frac{1}{|\mathbf{r}' - \mathbf{r}|} + \frac{-R/|\mathbf{r}' \cdot \mathbf{r}|}{|\mathbf{r} - \mathbf{r}'|} \right] ; \quad a' = \frac{R^2}{r'^2} \quad \frac{a'}{R} = \frac{R}{r'} \quad (3.15)
\]
**Point charge outside of a sphere at zero potential**

It should be noted that Eq. 3.15 provides the solution to the potential problem with a charge \( q \) located at \( r' \) **outside** or **inside** a spherical volume (where \( \frac{a}{R} = \frac{R}{r} \)) with the requirement that the potential of the sphere be zero. Likewise, Eq. 3.14a provides the Green’s function for both regions. Note that \( G_D (r, r') \) satisfies a Poisson equation where the charge density consists of two parts:

\[
\nabla^2 G_D (r, r') = \delta (r - r') - \frac{R}{r} \delta (a - r')
\]

(3.16)

The first term is non-zero (and the second term is zero) for \( r \) **inside** the volume of integration where the solution is defined. In other words, the Green’s function “image source” is outside the region for which you plan to find the solution.

**Example**: Find the potential, \( \phi (r) \) \( r > R \) with the constraint that \( \phi (r) = \phi_0 \) when \( |r| = R \).

**Solution**: In this case, the charge density, \( \rho (r') = 0 \), for \( r > R \), the region in which the solution is to be found. One can use the Green’s function, \( G_D (r, r') \), as given by Eq.3.14a. The volume integral is specified as \( |r| > R \) and the Green’s function has an “image charge \( q'' = -R/r \) located at \( a = (R/r)^2 \) **inside** the sphere of radius \( R \) centered at the origin. The Green’s function solution to this problem is (see Eq. 1.120)

\[
\phi_D (r) \bigg|_{r > R} = \frac{1}{\varepsilon} \int \int \int_{r' > R} G_D (r, r') \left[ -\rho (r') \right] d^3 r' + \int_{r' = R} \phi (r'_s) \nabla' G_D (r, r'_s) \cdot dS' - \int_{r' = R} G_D (r, r'_s) \nabla' \phi (r'_s) \cdot dS' + \int_{r' \to \infty} \phi (r'_s) \nabla' G_D (r, r'_s) \cdot dS' - \int_{r' \to \infty} G_D (r, r'_s) \nabla' \phi (r'_s) \cdot dS'
\]

The first integral on the right is 0 since \( \rho (r') = 0 \) for \( r' > R \). The third integral is 0 since \( G_D (r, r'_s) = 0 \) on the sphere. The two surface integrals at \( r' \to \infty \) are also 0 since both the \( G_D (r, r'_s) \) and the \( \phi (r'_s) \) approach 0 as \( r' \to \infty \) like \( 1/r' \). :

\[
\text{lim}_{r' \to \infty} \left[ \int \int_{r' \to \infty} \phi (r'_s) \nabla' G_D (r, r'_s) \cdot dS' - \int \int_{r' \to \infty} G_D (r, r'_s) \nabla' \phi (r'_s) \cdot dS' \right] \\
\leq \text{lim}_{r' \to \infty} \left[ c_1 \int \int_{r' \to \infty} \frac{1}{r'^2} \frac{1}{r'^2} \sin \theta' d\theta' d\phi' + c_2 \int \int_{r' \to \infty} \frac{1}{r'^2} \frac{1}{r'^2} r'^2 \sin \theta' d\theta' d\phi' \right] \\
= \text{lim}_{r' \to \infty} \left[ c_1 \frac{1}{r'} 4\pi + c_2 \frac{1}{r'} 4\pi \right] = 0
\]

where \( c_1 \) and \( c_2 \) are positive constants which = the maximal values of the integrands after the \( \frac{1}{r'} \frac{1}{r'} r'^2 \) is factored out.
Thus the solution is given by the second term which becomes:

$$\phi (r) \big|_{r>R} = \phi_0 \int_{r=R} \nabla' G_D (r, r') \cdot dS' \quad \text{where} \quad dS' = -\hat{r}' R^2 d\Omega'$$

The surface integral has the form \(\int \int -E_D \cdot dS'\) with \(E_D = -\nabla' G_D (r, r')\). and the Green's function can be viewed as a "potential" arising from a unit charge at \(r\) (outside the sphere) and an "image charge" \(Q_a = -\frac{R}{r}\) (inside the sphere) at \(a = \frac{R^2}{r}r\). Since \(E_D\) is evaluated just outside the sphere, the only charge which contributes to the surface integral is the "charge" enclosed, \(Q_a\). Note that the dependence of \(E_D\) on \(r\) and \(r'\) is non-trivial and we don’t need (or use) the explicit form of \(E_D\). Rather one uses Eq. 3.16 so that,

$$\int \int_{r'=R} -E_D \cdot dS' = \int \int_{r'=R} E_D \cdot -dS' = \int \int_{r'=R} E_D \cdot \hat{r}' R^2 d\Omega'$$

$$= \int \int_{r'=<R} \nabla' \cdot E_D dV' = \int \int_{r'=<R} \nabla'^2 G_D (r, r') dV'$$

$$= -\int \int \int_{r'<R} [\delta (r - r') - \frac{R}{r} \delta (a - r')] dV' = \frac{R}{r}$$

In this case the surface area is directed towards the origin and \(-\nabla' G_D (r, r')\) is the electric field due to the charges at \(r\) and \(a\). The surface integral expression for \(\phi (r)\) then becomes

$$\phi (r) = \phi_0 \frac{R}{r}$$

This is just the potential of a uniformly charged sphere with a charge of \(R\phi_0\).

**Solutions to Laplace’s equation: (x,y,z), (r,\theta, \varphi), (\rho, \varphi, z):**

**Helmholtz equation in \((r, \theta, \varphi)\)**

<table>
<thead>
<tr>
<th>Equation</th>
<th>separation const.</th>
<th>General solution: sum over all separation constants</th>
<th>conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\nabla^2 \Phi (x, y, z) = 0)</td>
<td>(k = k_1 \hat{x} + k_2 \hat{y} + k_3 \hat{z}); (k_3) real</td>
<td>(\sum_{k}\phi_0 [e^{i(k_1 x + k_2 y + k_3 z)} e^{ik_3 z} = &gt; \sin, \cos \text{ in two variables, exponential in third})</td>
<td>(k = k_0)</td>
</tr>
<tr>
<td>(\nabla^2 \Phi (r, \theta, \varphi) = 0)</td>
<td>(</td>
<td>m</td>
<td>\leq \ell = 0, 1, 2, \ldots)</td>
</tr>
<tr>
<td>(\nabla^2 \Phi (r, \theta, \varphi) = 0)</td>
<td>(</td>
<td>m</td>
<td>\leq \ell = 0, 1, 2, \ldots)</td>
</tr>
<tr>
<td>((\nabla^2 + k^2) \Phi (r, \theta, \varphi) = 0)</td>
<td>(</td>
<td>m</td>
<td>\leq \ell = 0, 1, 2, \ldots)</td>
</tr>
</tbody>
</table>

**Table 3.2**

<table>
<thead>
<tr>
<th>Function</th>
<th>separation const.</th>
<th>(e^{ik \cdot r})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Y_{\ell m}(\theta, \varphi))</td>
<td>(</td>
<td>m</td>
</tr>
<tr>
<td>(J_m(\alpha p))</td>
<td>(m = 0, \pm 1, \pm 2, \ldots)</td>
<td>(\sum_{j} (-1)^j \frac{\alpha p}{j!} \frac{2j + m}{2j!} \int_0^\infty \alpha p J_m(\alpha p) J_m(b p) d\rho = \delta (a - b); J_m(a) = J_m(b) = 0)</td>
</tr>
<tr>
<td>(j_\ell (kp); n_\ell (kp))</td>
<td>(\ell = 0, 1, 2)</td>
<td>(j_\ell (kp) = \frac{2 \pi}{2k' \rho} \frac{1}{J_{\ell + 1/2} (kp)}; \quad j_0 (x) = \frac{\sin x}{x}; \quad j_\ell (kp) \to \infty \to \frac{(kp)^\ell}{(2\ell + 1)!})</td>
</tr>
</tbody>
</table>

**boundary conditions:** the boundary conditions on the solution restrict the separation constants to specific values
3.2 Solutions of Laplace’s equation using series expansions

3.2.1 Laplace’s equation in a rectangular parallel-piped

One class of problems involves the solution to Laplace’s equation in a rectangular box. The potential is specified on each of the six rectangular surfaces of the box. An abbreviated version of these problems are the solutions to Laplace’s equation in a rectangle.

Example 1

Find the electrostatic potential inside the following volume, \( V: -a/2 \leq x \leq a/2, -b/2 \leq y \leq b/2; z \geq 0 \). Assume that there are no charges in the volume and \( \Phi(\pm a/2, y, z) = \Phi(x, \pm b/2, z) = \Phi_o, \ \Phi(x, y, z \to \infty) = 0, \) and \( \Phi(x, y, 0) = \Phi_o f(x, y) \).

Looking at the table on the previous page, we see that the general solution to \( \nabla^2 \Phi(x, y, z) = 0 \) is:

\[
\Phi(x, y, z) = \sum_{k \cdot -k=0} \left[ c(k) e^{k \cdot r} + d(k) e^{-k \cdot r} \right] ((ax + b)\delta_{k1,0} + (cy + d)\delta_{k2,0} + (ez + f)\delta_{k3,0})
\]

(Ex1_1)

1. First we note that the any term linear in \( x, y, \) or \( z \) solutions does not satisfy our boundary conditions so (see Table 3.1)

\[
\Phi(x, y, z) = \sum_{k \cdot -k=0} c(k) e^{k \cdot r} + C_o \quad \text{where} \quad k_1^2 + k_2^2 + k_3^2 = 0
\]

(Ex1-2)

The constant term, \( C_o \), however is not eliminated.

2. Since the solution must \( \to 0 \) as \( z \to \infty \) we take the solution periodic in \( x \) and \( y \) and exponential in \( z \). See Table 3.1

\[
\Phi(x, y, z) = \sum_k c(k) e^{i(k_1 x + k_2 y)} e^{-[|k_1|^2 + |k_2|^2]^{1/2} z} + C_o
\]

(Ex1-3)

3. Let \( (k_1, k_2) = (u, v) \) where \( u \) and \( v \) are real, positive numbers

\[
\Phi(x, y, z) = \sum_{u, v} c_{-}(u, v) [e^{iux} \pm e^{-iux}] [e^{ivy} \pm e^{-ivy}] e^{-[u^2 + v^2]^{1/2} z} + C_o
\]

4. Finally, since \( \Phi(\pm a/2, y, z) = \Phi(x, \pm b/2, z) = \Phi_o \) we must force the periodic terms vanish on the \( x \) and \( y \) boundaries and set \( C_o = \Phi_o \).

Let

\[
a/2 = m\pi \quad \text{where} \quad m = 1, 2, 3, \ldots \quad \text{and} \quad b/2 = n\pi \quad \text{where} \quad n = 1, 2, 3, \ldots
\]

\[
\Phi(x, y, z) = \sum_{m, n} c_{-}(m, n) \frac{1}{2i} \left[ e^{i m x} - e^{-i m x} \right] e^{2\pi m x} + e^{-2\pi m x}
\]

(Ex1-4)
Then,

\[
\Phi(x,y,z)=\sum_{m,n} c_{-(m,n)} \sin \left( \frac{2\pi x}{a} \right) \sin \left( \frac{2\pi n}{b} \right) e^{-2\pi[(x/a)^2+(y/b)^2]^{1/2}z} + \Phi_o
\]  

(Ex1-5)

5. To satisfy \( \Phi(x,y,0) = \Phi_o f(x,y) \) we note that at \( z=0 \),

\[
e^{-2\pi[(x/a)^2+(y/b)^2]^{1/2}} = 1 \quad \text{and}
\]

\[
\sum_{m,n} c_{-(m,n)} \sin \left( \frac{2\pi x}{a} \right) \sin \left( \frac{2\pi n}{b} \right) + \Phi_o = \Phi_o f(x,y)
\]

To determine the \( c_{\pm}(m,n) \) multiply both sides by \( \sin \left( \frac{2\pi x}{a} \right) \sin \left( \frac{2\pi n}{b} \right) \) and integrate over \( x \) and \( y \):

\[
\Phi_o \int \int \left[ f(x,y) - 1 \right] \sin \left( \frac{2\pi x}{a} \right) \sin \left( \frac{2\pi n}{b} \right) \, dx \, dy = \sum_{m,n} c_{-(m,n)} \int_{-\pi/2}^{\pi/2} \sin \left( \frac{2\pi x}{a} \right) \sin \left( \frac{2\pi n}{b} \right) \, dx \cdot \int_{-\pi/2}^{\pi/2} \sin \left( \frac{2\pi n'}{b} \right) \sin \left( \frac{2\pi n}{b} \right) \, dy
\]

\[
= \sum_{m,n} c_{-(m,n)} 2 \frac{a}{2\pi} \int_0^\pi \sin[mu] \sin[m'u] \, du \cdot \frac{b}{2\pi} \int_0^\pi \sin[nv] \sin[n'v] \, dv
\]

where \( u = \frac{2\pi}{a} x \) and \( v = \frac{2\pi}{b} y \). Using the orthogonality of the sin and cos functions,

\[\int_0^\pi \sin[mu] \sin[m'u] \, du = \frac{\pi}{2} \delta_{mm'}\]

and

\[\int_0^\pi \cos[mu] \cos[m'u] \, du = \frac{\pi}{2} \delta_{mm'}\]

(Ex1-6)

\[
= \sum_{m,n} c_{-(m,n)} \left[ \frac{2\pi}{a} \delta_{mm'} \cdot \frac{b}{2\pi} \delta_{nn'} \right] = c_{-(m',n')} \frac{ab}{4}
\]

\[
c_{-(m',n')} = \frac{4}{ab} \Phi_o \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \left[ f(x',y') - 1 \right] \sin \left( \frac{2\pi x'}{a} \right) \sin \left( \frac{2\pi n'}{b} \right) \, dx' \, dy'
\]

(Ex1-7a)

and

\[
\Phi(x,y,z) = \sum_{m,n} c_{-(m,n)} \sin \left( \frac{2\pi x}{a} \right) \sin \left( \frac{2\pi n}{b} \right) e^{-2\pi[(x/a)^2+(y/b)^2]^{1/2}z} + \Phi_o
\]

(Ex1-7b)
Section 3.2 Solutions of Laplace’s equation using series expansions

Laplace Equation in a rectangle

An abbreviated version of these problems are the solutions to the Laplace equation in a rectangle. The technique used for the two dimensional problem is readily generalized to the three dimensional rectangular parallel-piped.

Example 2 Boundary conditions given on a rectangle

Because the solutions to Laplace’s equations satisfy the superposition principle, it suffices to have \( \phi(x, y) = 0 \) zero on three of the four boundaries.

The function \( \phi(x, y) \) satisfies Laplace’s equation and the condition that \( \phi(0, y) = \phi(b, y) = \phi(x, 0) = 0 \) and \( \phi(x, a) = f(x) \). From Table 2.1 we look for a series solution of the form

\[
\phi(x, y) = \sum_{k} c(k)e^{\pm|k_1|x}e^{\pm k_2y} + C_o
\]

where each term in the series satisfies Laplace’s equation in two dimensions.

\[
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\phi(x, y) = 0
\]

\[
\sum_k [-|k_1|^2 + |k_2|^2]\phi(x, y) = 0
\]

In order to select the proper solution to these equations we refer to the boundary conditions. Since \( \phi(x, a) \) is specified in the boundary we use a (complete) set of orthogonal functions for the \( x \) dependence. Since \( \phi(0, y) = \phi(b, y) = 0 \) we use functions (from Table 2.1) which = 0 at \( y=0 \) and \( y=b \). It is also desirable to chose the \( y \) functions so that the factor reduces to 1 when \( y = a \). It follows that

\[
\phi(x, y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{b}\right) \frac{e^{\frac{\pi n}{b}y} - e^{-\frac{\pi n}{b}y}}{e^{\frac{\pi n}{b}a} - e^{-\frac{\pi n}{b}a}} = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{b}\right) \sinh\left(\frac{n\pi y}{b}\right) \sinh\left(\frac{n\pi a}{b}\right)
\]

with

\[
a_n = \frac{2}{b} \int_{0}^{b} f(x) \sin\left(n\pi x/b\right) \, dx
\]

As an example we take \( f(x) = \phi_0 \), a constant. In this case

\[
\alpha_n = \frac{2}{b} \int_{0}^{b} \phi_0 \sin\left(n\pi x/b\right) \, dx = \frac{2\phi_0}{n\pi} [1 - (-1)^n] \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{b}) = \phi_0 \quad \text{for} \quad \Pi X
\]
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and the potential is

\[ \phi(x, y) = \frac{4\phi_0}{\pi} \sum_{n=0}^{\infty} \frac{\sin((2n+1)\pi x/b) \sinh((2n+1)\pi y/b)}{2n+1} \frac{\sin((2n+1)\pi a/b)}{\sinh((2n+1)\pi a/b)} \]

For \( 0 \leq y < a \) this series and its first derivatives converge absolutely.
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Example 3  Boundary Conditions on a Semi-infinite strip

The region covered by the potential can also have an infinite extent. For example we let \( b \to \infty \), set \( \phi(x,0) = \phi(x,a) = 0 \), and \( \phi(0,y) = g(y) \).

For these boundary conditions the appropriate choice is

\[
\phi(x,y) = \sum_{n=1}^{\infty} \alpha_n \sin \left( \frac{n\pi y}{a} \right) \exp \left( -\frac{n\pi x}{a} \right)
\]

with

\[
\alpha_n = \frac{2}{a} \int_0^a g(y) \sin \left( \frac{n\pi y}{a} \right) dy.
\]

Example 4  In this example we let \( g(y) = V_0 y/a \). The expansion coefficients are given by

\[
\alpha_n = \frac{2V_0}{a^2} \int_0^a y \sin \left( \frac{n\pi y}{a} \right) dy
\]

\[
= \frac{2V_0}{a^2} \frac{a}{an\pi} \int_0^a y \cos \left( \frac{n\pi y}{a} \right) dy
\]

\[
= \frac{2V_0}{an\pi} \left[ -\frac{a}{n\pi} \cos \left( \frac{n\pi y}{a} \right) \right]_0^a - \int_0^a -\cos \left( \frac{n\pi y}{a} \right) dy
\]

\[
= -2 \left( \frac{(-1)^n V_0}{a\pi} \right).
\]

The potential is

\[
\phi(x,y) = -\frac{2V_0}{a^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \left( \frac{n\pi y}{a} \right) \exp \left( -\frac{n\pi x}{a} \right) \tag{Ex4-1}
\]

One can sum this series:

\[
\phi(x,y) = -\frac{2V_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left[ e^{-\frac{\pi y}{a}} - e^{-\frac{\pi y}{a}} \right] - \frac{1}{2i} \left[ e^{-\frac{\pi x}{a}} \right] \tag{Ex4-2}
\]

where \( u = e^{i\pi y/a}, v = e^{-\pi x/a} \)
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This series can be summed to yield

\[ \phi(x, y) = \frac{2V_0}{\pi} \arctan \left[ \frac{\exp(-\pi x/a) \sin(\pi y/a)}{1 + \exp(-\pi x/a) \cos(\pi y/a)} \right]. \]  

(Ex4-3)

Example 5

The last possible unique combination of boundary conditions again has \( b \to \infty \) but takes \( \phi(0, y) = \phi(x, 0) = 0 \) and \( \phi(x, a) = f(x) \).

In this case we switch from a summation to an integration. The expansion functions will be labelled by a continuous variable, say \( k \), for which the following relations are useful.

\[ \int_0^{\infty} \sin(kx) \sin(k'x) \, dx \]  

(Ex5-1)

\[ = \int_0^{\infty} \frac{1}{2i} [e^{ikx} - e^{-ikx}] \frac{1}{2i} [e^{ik'x} - e^{-ik'x}] \, dx \\
= \frac{-1}{4} \int_0^{\infty} [e^{ikx} - e^{-ikx}] [e^{ik'x} - e^{-ik'x}] \, dx \\
= \frac{-1}{4} \int_0^{\infty} [e^{ikx} e^{ik'x} + e^{-ikx} e^{-ik'x} - e^{ikx} e^{-ik'x} - e^{-ikx} e^{ik'x}] \, dx \\
= \frac{-2\pi}{4} \frac{1}{2\pi} \int_{-\infty}^{\infty} [e^{ikx} e^{ik'x} - e^{ikx} e^{-ik'x}] \, dx \\
= \frac{\pi}{2} \left[ \delta(k - k') - \delta(k + k') \right] \]
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The potential is given by

\[ \phi(x, y) = \int_0^\infty \alpha(k) \sin(kx) \frac{\sinh(ky)}{\sinh(ka)} \, dk \]

with

\[ f(x, a) = \int_0^\infty \alpha(k) \sin(kx) \, dk \]

\[ \int_0^\infty f(x) \sin(k'x) \, dx = \int_0^\infty \alpha(k) \left[ \int_0^\infty \sin(kx) \sin(k'x) \, dx \right] \, dk \]

\[ = \int_0^\infty \alpha(k) \frac{\pi}{2} \left[ \delta(k - k') + \delta(k + k') \right] \, dk \]

\[ \alpha(k) = \frac{2}{\pi} \int_0^\infty f(x) \sin(kx) \, dx. \]

As an example we take \( f(x) = V_0 \). The coefficients, given by the Fourier-sine transform of \( f(x) \), are

\[ \alpha(k) = \frac{2V_0}{\pi} \int_0^\infty \sin(kx) \, dx \]

\[ a(k) = \frac{2V_0}{\pi} \int_0^\infty \sin(kx) \, dx \]

\[ = \frac{2V_0}{\pi 2i} \int_0^\infty [e^{ikx} - e^{-ikx}] \, dx \]

an improper integral

To handle this integral we assume \( k \) is the imaginary part of a complex number, \( z = \epsilon + ik \), where \( \epsilon \) is real and small. Then,

\[ \alpha(k) = \frac{2V_0}{\pi 2i} \lim_{\epsilon \to 0} \lim_{b \to \infty} \left[ \int_0^b e^{-(\epsilon + ik)x} \, dx - \int_0^b e^{-(\epsilon - ik)x} \, dx \right] \]

\[ = \frac{2V_0}{\pi 2i} \lim_{\epsilon \to 0} \left[ \frac{1}{\epsilon - ik} - \frac{1}{\epsilon + ik} \right] \]

\[ = \frac{2V_0}{\pi 2i} \lim_{\epsilon \to 0} \left[ \frac{\epsilon - \epsilon' + 2ik}{\epsilon \epsilon' + k^2 - ik(\epsilon' - \epsilon)} \right] \]

\[ = \frac{2V_0}{\pi k} \]

Note that this result can be used in general:

\[ \int_0^\infty e^{-a_0 x} \, dx = \frac{1}{a_0} \text{ if } a_0 > 0 \]  \hspace{1cm} (Ex5-2)

using this result for \( \alpha(k) \) the potential is given by

\[ \phi(x, y) = \frac{2V_0}{\pi} \int_0^\infty \sin(kx) \frac{\sinh(ky)}{\sinh(ka)} \, dk \]  \hspace{1cm} (Ex5-3)

\[ = \frac{V_0}{\pi} \int_{-\infty}^{\infty} \sin(kx) \frac{\sinh(ky)}{\sinh(ka)} \, dk \]

We shall evaluate this integral using the theory of residues.
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Assume an analytic function $F(z)$ has an $m^{th}$ order pole at $z = z_0$. Then if $F(z)$ is integrated around a contour enclosing $z_0$ we obtain the residue of $F(z)$ at $z = z_0$,

$$
\int F(z) \, dz = 2\pi i \, res\left(z_0\right)
$$

(Ex5-4)

In the case of an $m^{th}$ order pole

$$
res\left(z_0\right) = \frac{1}{(m - 1)!} \left[ \frac{d^{m-1}\left((z - z_0)^m F(z)\right)}{dz^{m-1}} \right]_{z=z_0}
$$

(Ex5-5)

Note: the residue is always the factor multiplying $\frac{1}{z - z_0}$ in the expansion of $F(z)$, (independent of how many other terms of the form, $\frac{1}{(z - z_0)^m}$, there are in the expansion of $F(z)$)

As it stands the integral in Eq. Ex5-3 (where we assume $k$ is complex) has no poles along the real $k$ axis. So we can let $k = k - i\epsilon$ and later let $\epsilon \rightarrow 0^+$. Note that if we can show that the semi-circular contour, $C_{semi-circle}$ at $|k| \rightarrow \infty$ vanishes, we can set

$$
\phi(x, y) = \frac{V_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin(kx) \sinh(\epsilon k)}{k - i\epsilon \sinh(\epsilon k)} \, dk + \frac{V_0}{\pi} \int_{C_{semi-circle}} \frac{\sin(kx) \sinh(\epsilon k)}{k - i\epsilon \sinh(\epsilon k)} \, dk
$$

(Ex5-5b)

$$
\phi(x, y) = \frac{V_0}{2i\pi} \int_{C_{cloud}} \frac{\exp(ikx) - \exp(-ikx) \sinh(k\epsilon)}{k - i\epsilon \sinh(\epsilon k)} \, dk
$$

The contour will be closed in the upper half complex plane ($C_+$ in the first integral) when using $\exp(ikx)$ and in the lower half plane ($C_-$ in the second integral) when using $\exp(-ikx)$. Note that the contours are chosen so that at $|k| \rightarrow \infty$ the integrands converge like $\exp(-|k|x)$. The $C_-$ is traversed in the “negative” direction, giving an overall minus sign to the residue from the lower half plane contour.

The poles of the integrand come from:

(1) the $\frac{1}{k - i\epsilon}$ factor (at $k = 0 + i\epsilon$ with $\epsilon \rightarrow 0^+$);

The residue from this pole is

$$
\lim_{\epsilon \rightarrow 0} \left[ \frac{V_0}{2i\pi} \frac{\exp(i\epsilon x) \sinh(i\epsilon y)}{1 \sinh(i\epsilon a)} \right] = \lim_{\epsilon \rightarrow 0} \left[ \frac{V_0}{2i\pi} \frac{1}{\frac{d}{dx} \sinh(\epsilon y)_{\epsilon=0}} \right]
$$

$$
= \lim_{\epsilon \rightarrow 0} \left[ \frac{V_0}{2i\pi} \frac{1}{\frac{d}{dx} \sinh(\epsilon a)_{\epsilon=0}} \right]
$$

$$
= \lim_{\epsilon \rightarrow 0} \left[ \frac{V_0}{2i\pi} \frac{1}{\frac{d}{dx} \cos(\epsilon a)_{\epsilon=0}} \right]
$$

$$
= \frac{V_0 y}{2i\pi a}
$$

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Note that this pole is only in the $C_+$ contour.

2) the $\frac{1}{\sinh(ka)} = \frac{1}{\sinh(in\pi)} = \frac{1}{\sinh(n\pi)}$ factor at $ka = n\pi i$ of the integrand in the upper half plane. The $\sinh(ka)$ can be expanded near $ka = n\pi i$ so that the structure of the pole is

$$\frac{1}{\sinh(ka)} \approx \frac{1}{\sinh(in\pi) + \sinh(in\pi) \frac{[ka - in\pi]^2}{2} + ...}$$

$$= \frac{1}{i \sin(n\pi) + \cosh(in\pi)[ka - in\pi] + \sinh(in\pi) \frac{[ka - in\pi]^2}{2} + ...}$$

$$= \frac{1}{a \cos(n\pi)[k - in\pi/a] + 0 + ..}$$

and has the same residue as a simple pole at $k = n\pi i/a$. When the integrand is multiplied by $[k - in\pi/a]$ and the limit taken as $ka \rightarrow in\pi$ we have for the residue

$$\frac{V_0}{2\pi} (-1)^n \exp(-i \cdot in\pi x/a) \sinh(in\pi y/a)$$

$$= \frac{V_0}{2\pi} (-1)^n \exp(-n\pi x/a) \sin(n\pi y/a)$$

(3) Similarly the $\frac{1}{\sinh(ka)}$ factor at $ka = -n\pi i$ in the lower half plane gives a residue (note that the contour is along the "negative" direction in the lower half plane),

$$(-1)^n \left[ \frac{V_0}{2\pi i} (-1)^n \exp(-i \cdot -in\pi x/a) \sin(-in\pi y/a) \right]$$

$$= -\frac{V_0}{2\pi i} (-1)^n \exp(-n\pi x/a) \sin(-n\pi y/a)$$

$$= \frac{V_0}{2\pi i} (-1)^n \exp(-n\pi x/a) \sin(n\pi y/a)$$

The potential is obtained from the sum over all the residues

$$\phi(x, y) = 2\pi i \Sigma_j \text{Residues}(z_{oj})$$

$$= \frac{V_0 y}{a} + \frac{2V_0}{\pi} \sum_{n=1}^{\infty} (-1)^n \exp(-n\pi x/a) \sin(n\pi y/a).$$
The function \( \phi(x, y) \) satisfies Laplace’s equation and the condition that \( \phi(x, a) = V_0, \phi(x, 0) = 0 \) and \( \phi(0, y) = V_0 y/a \). Any superposition of solutions to Examples 2, 4 and 5 is also a solution to Laplace’s equation. If we add the solutions to Examples 4 and 5 we obtain the following:

\[
\phi_{4+5}(x, y) = \frac{V_0 y}{a} + \frac{2V_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \exp \left( -n\pi x/a \right) \sin \left( n\pi y/a \right) - \frac{2V_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \left( n\pi y/a \right) \exp \left( -n\pi x/a \right)
\]

\[
= \frac{V_0 y}{a}
\]
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The rectangle is still infinite in length along x, but \( \phi_{4+5} (x, y) \) has no x dependence.

**Example 6. Poisson’s equation in a wedge shaped region**

The figure below shows the wedge shaped region lying between \( \rho = a \) and \( \rho = b \) in radius and between the angles \( \Theta = 0 \) and \( \Theta = \beta \).

Laplace’s equation in this coordinate system is given by:

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \phi (\rho, \theta)}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \phi (\rho, \theta)}{\partial \theta^2} = 0
\]  

A solution of this equation, which vanishes at \( \theta = 0 \) and \( \theta = \beta \), can be expressed as

\[
\phi (\rho, \theta) = \sum_{n=1}^{\infty} \left( a_n \rho^{n\pi/\beta} + b_n \rho^{-n\pi/\beta} \right) \sin \left( \frac{n\pi \theta}{\beta} \right).
\]

If \( \phi (a, \theta) = f_1 (\theta) \) and \( \phi (b, \theta) = f_2 (\theta) \) then

\[
a_n a^{n\pi/\beta} + b_n a^{-n\pi/\beta} = \frac{2}{\beta} \int_0^\beta \sin \left( \frac{n\pi \theta'}{\beta} \right) f_1 (\theta') \, d\theta'
\]
\[
a_n b^{n\pi/\beta} + b_n b^{-n\pi/\beta} = \frac{2}{\beta} \int_0^\beta \sin \left( \frac{n\pi \theta'}{\beta} \right) f_2 (\theta') \, d\theta'
\]

In the case that \( b \to \infty \) and \( \phi (\rho, \theta) \to 0 \) for \( \rho \to \infty \) the coefficients \( a_n = 0 \) and

\[
b_n a^{-n\pi/\beta} = \frac{2}{\beta} \int_0^\beta \sin \left( \frac{n\pi \theta'}{\beta} \right) f_1 (\theta') \, d\theta.
\]

Conversely, if \( a \to 0 \) and \( \phi (0, \theta) \) is finite, then the coefficients \( b_n = 0 \) and

\[
a_n b^{n\pi/\beta} = \frac{2}{\beta} \int_0^\beta \sin \left( \frac{n\pi \theta'}{\beta} \right) f_2 (\theta') \, d\theta'
\]
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Special problems

1. Using the Green’s function for the Dirichlet problem, find $\phi(r)$ inside a grounded, conducting sphere of radius, $R$, for the case where there is a charge located at $r = \frac{R}{3}\hat{x}$.

2. Using the Green’s function for the Dirichlet problem, find $\phi(r)$ outside a grounded, conducting sphere of radius, $R$, for the case where there is a charge located at $r = 2R\hat{x}$. 
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3. Using the Green’s function for the Dirichet problem, find $\phi(r)$ **inside** a conducting sphere of radius, $R$, for the case where $\nabla^2 \phi(r) = 0$ everywhere and $\phi(R\hat{r}) = \frac{Q}{R}$.

4. Using the Green’s function for the Dirichet problem, find $\phi(r)$ **outside** a conducting sphere of radius, $R$, for the case where $\nabla^2 \phi(r) = 0$ everywhere and $\phi(R\hat{r}) = \frac{Q}{R}$.
5. Consider an electric field problem in the half space \( z \geq 0 \) with Neumann boundary condition that \( E_z(x, y, 0) = E_0 \) for \(|x| < w/2\) and \( E_z(x, y, 0) = 0 \) V/m otherwise. (a) Show that \( E_y(x, y, z) = 0 \) V/m. (b) Evaluate \( E_z(x, y, z) \) and make a contour plot of \( E_z(x, y, z)/E_0 \) versus \( x/w \) and \( z/w \) for \( 0 < x/w < 3 \) and \( 0 < z/w < 2 \). (c) Taking the potential equal to zero at \((0, 0, 0)\) make a contour plot of the \( \phi(r)/E_0w \) in the over the same range of values used in parts (b) and (c).

6. A sphere of radius 1 cm sits 2 cm above the center of a large grounded conducting plane. As a model assume that the plane is infinite in extent. Choosing a coordinate system with the sphere centered at \( z = 2 \) cm and the plane at \( z = 0 \) cm use the Dirichlet Green’s function for the region \( z \geq 0 \) cm and the and the surface charge density \( \sigma(z) = \sigma_0 + b(z - 2 \text{ cm}) \) to obtain a lower bound on the capacitance of the sphere.
3.2.2 Some useful relations between path integrals

Many expressions involve principal value integrals. For these integrals the integrand will have simple poles on the real axis and the integral is taken along the real axis excluding the location of the pole.

The principal value integral

equals one half of

\[ P \int_{-\infty}^{\infty} \frac{f(x)}{x} \, dx = \lim_{\delta \to 0} \left[ \int_{-\infty}^{-\delta} \frac{f(x)}{x} \, dx + \int_{\delta}^{\infty} \frac{f(x)}{x} \, dx \right] \] (A3-1)

Often it is easier to evaluate the integral using Cauchy’s residue theorem. This requires a closed path and the principal value integration leaves a gap in the path. This problem is surmounted using the identity illustrated in Fig. 3A-1. The integrand is assumed to have a simple pole located at the ‘x’ on the real axis. The identity holds because one can show that the integrals along the half circles in the two contours cancel. In this case

\[ P \int_{-\infty}^{\infty} \frac{f(x)}{x} \, dx = \frac{1}{2} \left[ \int_{-\infty}^{\infty} \frac{f(x)}{x} \, dx + \int_{-\infty}^{\infty} \frac{f(x)}{x} \, dx \right] \] (A3-2)

The principal value integral has been replaced by two integrals but Cauchy’s residue theorem can often be used to evaluate each integral.
Another relationship between the path integrals is often useful. This identity is illustrated in Fig. 3A-2 where again ‘x’ marks the location of the pole. The direction of integration along the circular path around the pole has been selected so as to cancel the half circle for $C_+$ and add the half circle for $C_-$. If both identities are combined the principal value integral is seen to equal (1) the integral along path $C_+$ plus one half the integral around the pole or (2) the integral along the path $C_-$ minus one half the integral around the pole.