Problem Set 3  
(Jackson 6.20).

1. An example of the preservation of causality and finite speed of propagation in spite of the use of the Coulomb gauge is afforded by a unit strength dipole source that is flashed on and off at \( t = 0 \). The charge and current densities are

\[
\rho(\mathbf{r}, t) = \delta(x)\delta(y)\delta'(z)\delta(t)
\]

\[
\mathbf{J}(\mathbf{r}, t) = -e_3 \delta(x)\delta(y)\delta(z)\delta'(t)
\]

where a prime means differentiation with respect to the argument. This dipole is of unit strength and it points in the negative \( z \) direction.

(a) Show that the instantaneous Coulomb potential (6.23 in text) is

\[
\phi(\mathbf{r}, t) = -\frac{1}{4\pi \varepsilon_0} \frac{\hat{z}}{r} \delta(t).
\]

(b) Show that the transverse current, \( \mathbf{J}_t \), is

\[
\mathbf{J}_t(\mathbf{r}, t) = -\left[ e_3 \frac{2}{3} \delta^{(3)}(\mathbf{r}) + \frac{1}{4\pi} \frac{3\mathbf{n} \cdot e_3 - e_3}{r^3} \right] \delta'(t)
\]

where \( \mathbf{n} = \hat{r} \), a unit vector along the \( \mathbf{r} \) direction and the \( \frac{2}{3} \) factor multiplying the delta function comes from treating the gradient of \( \frac{1}{r^3} \) according to (4.20 in text.)

(c) Show that the electric and magnetic fields are causal and that the electric field components are given by:

\[
\mathbf{E}(\mathbf{r}, t) = e_3 \frac{c}{r} \left[ \delta''(ct - r) + \frac{1}{r^2} \delta(r - ct) - \frac{1}{r} \delta'(r - ct) \right]
\]

\[
- \mathbf{r} \frac{c\varepsilon}{r^3} \left[ \delta''(r - ct) - \frac{3}{r} \delta'(r - ct) + \frac{3}{r^2} \delta(r - ct) \right]
\]

Hint: While the answer in part b displays the transverse current explicitly, the less explicit form,

\[
\mathbf{J}_t(\mathbf{r}, t) = -e_3 \delta(\mathbf{r})\delta'(t) - \delta(t) \frac{1}{4\pi} \frac{\partial}{\partial z} \left( \nabla \frac{1}{r} \right)
\]

can be used with (6.47 in text) to calculate the vector potential and the fields for part c. An alternative method is to use the Fourier transforms in time of \( \mathbf{J}_t \) and \( \mathbf{A} \), the Green function (6.40) and its spherical wave expansion from Chapter 9.

Solution:

Discussion: The scalar potential will equal the that of a static point dipole multiplied by \( \delta(t) \). Its contribution to the electric field will be a dipole field multiplied by \( \delta(t) \). This contribution must be cancelled by a contribution from the vector potential. This requires that a term multiplied by \( \delta(t) \) must be obtained from the vector potential. There is not a unique procedure by which this can be achieved. In the approach used here the time derivative of the vector potential will be carried out before the space integration is completed.

Solution (a):

In the coulomb gauge the scalar potential satisfies Poisson’s equation with the charge density as the source. The scalar potential is obtained using the Green’s function for Poisson’s equation
\[
\phi(r, t) = \iiint \frac{\delta(x')\delta(y')\delta'(z')\delta(t)}{|r - r'|} \, d^3r' \\
= \delta(t) \frac{\partial}{\partial z} \left( \frac{1}{r} \right) = -\delta(t) \frac{z}{r^3} \quad \text{Gaussian units} \\
= \frac{1}{4\pi\varepsilon_0} \delta(t) \frac{\partial}{\partial z} \left( \frac{1}{r} \right) \quad \text{SI units}
\]

The spatial dependence is that of the unit electric dipole \( p = -e_3 \).

**Solution (b):** The vector potential satisfies the wave equation with the transverse current acting as the source. The transverse current density is most easily obtained by subtracting the longitudinal current density from the current density. From the lecture notes the longitudinal current density is given by

\[
\mathbf{J}_l(r, t) = -\frac{1}{4\pi} \mathbf{V} \left[ \frac{\partial}{\partial t} \phi(r, t) \right] \\
= -\frac{1}{4\pi} \delta'(t) \mathbf{V} \left( \frac{\partial}{\partial z} \frac{1}{r} \right)
\]

We recognize that the longitudinal current density has the same spatial dependence as the electric field due to a dipole \( p = +e_3 \) at the origin. This field is

\[
\mathbf{E}_{\text{dipole}}(r) = -\frac{4\pi}{3} \delta^{(3)}(r) e_3 + \frac{3\mathbf{n} \cdot e_3}{r^3} - e_3
\]

The transverse current density will be

\[
\mathbf{J}_t(r, t) = -e_3 \delta(x)\delta(y)\delta(z)\delta'(t) - \delta'(t) \frac{1}{4\pi} \nabla \cdot \frac{\partial}{\partial z} \frac{1}{r} \\
= -e_3 \delta(x)\delta(y)\delta(z)\delta'(t) - \delta'(t) \left[ -\frac{1}{4\pi} \left( \nabla \frac{-z}{r^3} + e_3 \frac{\partial^2}{\partial z^2} \frac{1}{r} \right) \right]_{|r| = 0}
\]

\[
= -\left[ e_3 \frac{\partial}{\partial r} + \frac{1}{4\pi} (-e_3 + 3(\mathbf{e}_3 \cdot \hat{r})) \frac{1}{r} - e_3 \frac{\partial}{\partial z} \right] \delta'(t)
\]

\[
= -e_3 \frac{2}{3} \delta^{(3)}(r) - \frac{1}{4\pi} \frac{3\mathbf{n} \cdot e_3}{r^3} - e_3
\]

**Solution (c):** The vector potential can be evaluated using the retarded or causal Green’s function for the wave equation

\[
\mathbf{A}(r, t) = -\frac{1}{c} \iiint \frac{\delta(t - t' - c^{-1}|r - r'|)}{|r - r'|} \left[ e_3 \delta^{(3)}(r') + \frac{1}{4\pi} \frac{\partial}{\partial z'} \left( \nabla' \frac{1}{r'} \right) \right] \delta'(t') \, d^3r' \, dt'
\]

\[
= -e_3 \frac{\delta'(t - r/c)}{r} - \frac{1}{4\pi c} \iiint \frac{\delta'(t - c^{-1}|r - r'|)}{|r - r'|} \frac{\partial}{\partial z'} \left( \nabla' \frac{1}{r'} \right) \, d^3r' \\
= -\frac{e_3}{c} \frac{\delta'(t - r/c)}{r} - \frac{1}{4\pi c} \iiint \frac{\delta'(t - R/c)}{R} \frac{\partial}{\partial Z} \left( \nabla_R \frac{1}{|R - r|} \right) \, d^3R;
\]
Note: \( \mathbf{R} = \mathbf{r} - \mathbf{r}' \) \( \Rightarrow |\mathbf{r}'| = |\mathbf{r} - \mathbf{R}| = |\mathbf{R} - \mathbf{r}| \) and \( \nabla \mathbf{R} \frac{1}{|\mathbf{R} - \mathbf{r}|} = -\nabla \frac{1}{|\mathbf{R} - \mathbf{r}|} \)

\[
\mathbf{A}(\mathbf{r}, t) = -\frac{\mathbf{e}_3}{c} \frac{\delta'(t - r/c)}{r} - \frac{1}{4\pi c} \frac{\partial}{\partial z} \nabla \int \int \int \int \frac{\delta'(t - R/c)}{R} \frac{1}{|\mathbf{R} - \mathbf{r}|} d^3 R
\]

In this form the vector potential exhibits a discontinuity at \( t = 0 \). For \( t < 0 \) \( \mathbf{A}(\mathbf{r}, t) = 0 \) and for \( t > 0 \) \( \mathbf{A}(\mathbf{r}, t) \neq 0 \). This discontinuity must be the source of the term involving \( \delta(t) \). The time dependence of the second term is contained in the integral. To take the time derivative of the second term we analyze

\[
V(\mathbf{r}, t) = \frac{1}{4\pi c} \int \int \int \frac{\delta'(t - R/c)}{R} \frac{1}{|\mathbf{R} - \mathbf{r}|} d^3 R; \text{ expand the } \frac{1}{|\mathbf{R} - \mathbf{r}|} \text{ in spherical harmonics.}
\]

\[
= \frac{1}{4\pi c} \int \int \int \frac{\delta'(t - R/c)}{R} \sqrt{4\pi} Y_{00}(\theta', \phi') \sum_{l,m} \frac{4\pi}{2l + 1} \frac{r^l_p}{r^l_m} Y_{lm}(\theta', \phi') Y_{lm}(\theta, \phi) R^2 dR \sin \theta' d\theta' d\phi'
\]

\[
= \frac{1}{c} \int_0^r \delta'(t - \frac{R}{c}) \frac{R}{r} dR + \int_r^\infty \delta'(t - \frac{R}{c}) dR \quad \text{only the } l, m = 0, 0 \text{ term contributes.}
\]

\[
= \frac{c}{r} \int_0^r \delta'(R - ct) R dR + c \int_r^\infty \delta'(R - ct) dR
\]

Note that \( \delta'(ct) = \frac{d}{dt} \delta(ct) \):

\[
\int \delta'(ct)f(t) dt = \int \frac{d}{dt} \delta(ct)f(t) dt
\]

\[
= \frac{1}{c} \int \frac{d}{dt} \delta(ct)f(t) dt
\]

\[
= \frac{1}{c^2} \int \delta'(t)f(t) dt
\]

\[
= \frac{1}{c^2} \int \delta'(t)f(t) dt
\]

The time derivative of this term is

\[
\frac{\partial}{\partial t} V(\mathbf{r}, t) = \frac{c^2}{r} \int_0^r \delta''(R - ct) R dR + c^2 \int_r^\infty \delta''(R - ct) dR
\]

\[
= \frac{c^2}{r} [R\delta'(R - ct)|_0^b - \int_0^r \delta'(R - ct) dR] + c^2 \delta'(R - ct)|_r^\infty
\]

Note: \( \int_a^b \delta'(x) dx = \delta'(x)|_a^b \) and \( \int_a^b \delta''(x) dx = \delta(x)|_a^b \)

\[
= \frac{c^2}{r} [r\delta'(r - ct) - \int_0^r \delta'(R - ct) dR] + c^2 \lim_{R \to \infty} \delta'(R - ct) - c^2 \delta'(r - ct)
\]

\[
= \frac{c^2}{r} [-\delta(R - ct)|_0^b \quad \text{for finite values of } t
\]

\[
= \frac{c^2}{r} [\delta(ct) - \delta(r - ct)] = \frac{c}{r} \delta(t) - \frac{c}{r} \delta(t - \frac{r}{c})
\]

The contribution of the vector potential to the electric field, \( \mathbf{E}(\mathbf{r}, t) = -\nabla \Phi - \frac{1}{c} \frac{\partial}{\partial t} \mathbf{A} \), is

\[
- \frac{1}{c} \frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{e}_3}{c^2} \frac{\delta''(t - r/c)}{r} + \frac{\partial}{\partial z} \nabla \left[ \frac{1}{r} \delta(t) - \frac{1}{r} \delta(t - \frac{r}{c}) \right]
\]
while the scalar potential gives

\[- \nabla \phi(r, t) = -\delta(t) \frac{\partial}{\partial z} \nabla \left( \frac{1}{r} \right) .\]

The electric field is the sum of these

\[E(r, t) = \frac{e_3}{c^2} \delta''(t - r/c) r - \frac{\partial}{\partial z} \nabla \left[ \frac{1}{r} \delta\left(t - \frac{r}{c}\right) \right] \]

\[= \frac{e_3}{c^2} \delta''(ct - r) r - c \frac{\partial}{\partial z} \nabla \left[ \frac{1}{r} \delta(r - ct) \right] \]

\[= \frac{e_3}{c^2} \delta''(ct - r) r - c \nabla \left[ \left( \frac{z}{r} \right) \frac{\partial}{\partial r} \left\{ \frac{1}{r} \delta(r - ct) \right\} \right] \]

\[= \frac{e_3}{c^2} \delta''(ct - r) r - \frac{\partial}{\partial r} \left\{ \frac{1}{r} \delta(r - ct) \right\} \]

\[= -cz \frac{r}{r} \frac{\partial}{\partial r} \left[ \left( \frac{1}{r} \right) \frac{\partial}{\partial r} \left\{ \frac{1}{r} \delta(r - ct) \right\} \right] \]

At this point we note that the electric field propagates out from the dipole with the speed of light. The electric field is.

\[E(r, t) = \frac{e_3}{c^2} \delta''(ct - r) + \frac{1}{r^2} \delta(r - ct) - \frac{1}{r} \delta'(r - ct) \]

\[= -r \frac{cz}{r^3} \delta''(r - ct) - \frac{3}{r^2} \delta'(r - ct) + \frac{3}{r^2} \delta(r - ct) \]

\[= \left[ \frac{e_3}{r^3} - r \frac{e_3}{r^3} \right] \delta(r - ct) + \text{time derivative terms.} \]

Note: Since \( \nabla \times \nabla f(x, y, z) = 0 \) the magnetic flux density is found from

\[\nabla \times E = \nabla \times \left( \frac{e_3}{c^2} \frac{\delta''(t - r/c)}{r} \right) = -\frac{1}{c} \frac{\partial}{\partial t} [\nabla \times \left( \frac{e_3}{c} \frac{\delta'(t - r/c)}{r} \right)] \quad \text{and} \quad \text{#} \]

\[B(r, t) = \nabla \times \left( \frac{e_3}{c} \frac{\delta'(t - r/c)}{r} \right) \quad \text{#} \]

\[= -e_3 \times \hat{r} \left[ \frac{\partial}{\partial r} \delta'(t - r/c) \right] \quad \text{#} \]

which propagates outward with the speed of light in the xy plane. The derivatives of the Dirac Delta functions are only "non-zero" at \( r = tc \) as everywhere else the \( \delta(r - ct) = 0 \).
2. Using the retarded Green’s function and the Lorentz gauge, find the electrostatic potential, \( \Phi(\mathbf{r}, t) \), generated by a point charge, \( Q \), moving with velocity \( \mathbf{v}_o \). Assume \( \Phi(\mathbf{r}, t) \to 0 \) and \( \frac{\partial \Phi(\mathbf{r}, t)}{\partial t} \to 0 \) as \( t \to -\infty \). Hint: Do the space integration first and recall the definition of the Dirac delta function whose argument is a function.

You might also find it useful to use the following expressions to simplify the notation:
\( \tau \equiv ct \), \( \tau' \equiv ct' \), \( \mathbf{\beta} \equiv \mathbf{v}_o / c \), \( \mathbf{R} \equiv \mathbf{r} - \mathbf{\beta} \tau = \mathbf{r} - \mathbf{v}_o t \), \( W \equiv \tau - \tau' \); Note that \( |\mathbf{R} + \mathbf{\beta} W| = |\mathbf{r} - \mathbf{v}_o t'| \)

**Solution:**
1. The wave equation for the electrostatic potential is
\[ \nabla^2 \Phi(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \Phi(\mathbf{r}, t)}{\partial t^2} = -4\pi \rho(\mathbf{r}, t) \] with \( \rho(\mathbf{r}, t) = Q \delta(\mathbf{r} - \mathbf{v}_o t) \). The Green’s function solution is given by:
\[
\Phi(\mathbf{r}, t) = \iiint G_R(\mathbf{r} - \mathbf{r}', t - t')\rho(\mathbf{r}', t')d^3r'dt'
= \iiint \Theta(t - t') \frac{c}{|\mathbf{r} - \mathbf{r}'|} \delta(|\mathbf{r} - \mathbf{r}'| - c(t - t'))Q\delta(\mathbf{r}' - \mathbf{v}_o t')d^3r'dt'
= \int \Theta(t - t') \frac{c}{|\mathbf{r} - \mathbf{v}_o t'|} \delta(|\mathbf{r} - \mathbf{v}_o t'| - c(t - t'))Q dt'
= Q \int \Theta(t - t') \frac{c}{|\mathbf{r} - \mathbf{v}_o t'|} \delta(|\mathbf{r} - \mathbf{v}_o t'| - c(t - t'))dt'
\]

2. To do the \( t' \) integration convert the \( \delta(|\mathbf{r} - \mathbf{v}_o t'| - c(t - t')) \) using the expression \( \delta(f(x)) = \sum_i \delta(x - x_{oi}) / |f'(x)|_{x=x_{oi}} \).

First, find the zeros of the argument using \( W \equiv c(t - t') = \tau - \tau' \):
\[
|\mathbf{r} - \mathbf{v}_o t'| - c(t - t') = 0
|\mathbf{r} - \mathbf{\beta} \tau + \mathbf{\beta}(\tau - \tau')| - (\tau - \tau') = 0
|\mathbf{R} + \mathbf{\beta} W| - W = 0
(\mathbf{R} + \mathbf{\beta} W) \cdot (\mathbf{R} + \mathbf{\beta} W) = W^2
R^2 + 2\mathbf{R} \cdot \mathbf{\beta} W + W^2(\beta^2 - 1) = 0
W^2 - 2\mathbf{R} \cdot \mathbf{\beta}^2 W - R^2\gamma^2 = 0
\]
where \( \gamma^2 = 1/(1 - \beta^2) \). The two solutions are:
\[ W_+ = \gamma^2 \mathbf{R} \cdot \mathbf{\beta} [1 \pm \sqrt{1 + [\frac{R}{\gamma(\mathbf{R} \cdot \mathbf{\beta})}]^2}] \]
\[ ct - ct'_+ = \gamma^2 \mathbf{R} \cdot \mathbf{\beta} [1 + \sqrt{1 + [\frac{R}{\gamma(\mathbf{R} \cdot \mathbf{\beta})}]^2}] \]

since only \( t > t' \) is allowed by causality.

3. Actually, we will find that the solution is independent of the choice of sign. Now evaluate the Dirac delta function:
\[
\delta(|\mathbf{r} - \mathbf{v}_o t'| - c(t - t')) = \delta(W - W_+)|\frac{\partial}{\partial W} ([(R + \beta W) - \mathcal{W}] |_{W = W_+})
\]

\[
= \delta(W - W_+)|\frac{\partial}{\partial W} ([(R + \beta W) - \mathcal{W}] |_{W = W_+})
\]

\[
= \delta(ct' - ct_+) |\frac{\partial}{\partial W} ([(R + \beta W) - \mathcal{W}] |_{W = W_+})
\]

\[
= \frac{1}{c} \delta(t' - t_+) |\frac{\partial}{\partial W} ([(R + \beta W) - \mathcal{W}] |_{W = W_+})
\]

4. The integral becomes

\[
\Phi(\mathbf{r}, t) = Q \int \Theta(t - t') \frac{1}{|\mathbf{R} + \beta \mathcal{W}|} \delta(t' - t_+) |\frac{\partial}{\partial W} ([(R + \beta W) - \mathcal{W}] |_{W = W_+}) dt'
\]

\[
= Q \int \Theta(t - t') \frac{1}{|\mathbf{R} + \beta \mathcal{W}|} |\mathbf{R} + \beta \mathcal{W}| \delta(W - W_+) |\frac{\partial}{\partial W} ([(R + \beta W) - \mathcal{W}] |_{W = W_+}) dW
\]

\[
= Q \int \Theta(t - t') \frac{1}{|\mathbf{R} + \beta \mathcal{W}|} \frac{1}{\mathbf{R} \cdot \beta} \frac{1}{|\mathbf{R} + \beta \mathcal{W}|} \mathcal{W} dW
\]

\[
= Q \int \Theta(t - t') \frac{1}{|\mathbf{R} \cdot \beta| \mathcal{W}} \frac{1}{\sqrt{1 + \left[ \mathbf{R} / \gamma(\mathbf{R} \cdot \beta) \right]^2}} dW
\]

\[
= Q \int \Theta(t - t') \frac{1}{|\mathbf{R} \cdot \beta| \mathcal{W}} \frac{1}{\sqrt{1 + \left[ \mathbf{R} / \gamma(\mathbf{R} \cdot \beta) \right]^2}} dW
\]

\[
= Q \int \Theta(t - t') \frac{1}{|\mathbf{R} \cdot \beta| \mathcal{W}} \frac{1}{\sqrt{1 + \left[ \mathbf{R} / \gamma(\mathbf{R} \cdot \beta) \right]^2}} dW
\]

\[
= Q \int \Theta(t - t') \frac{1}{|\mathbf{R} \cdot \beta| \mathcal{W}} \frac{1}{\sqrt{1 + \left[ \mathbf{R} / \gamma(\mathbf{R} \cdot \beta) \right]^2}} dW
\]

\[
= Q \int \Theta(t - t') \frac{1}{|\mathbf{R} \cdot \beta| \mathcal{W}} \frac{1}{\sqrt{1 + \left[ \mathbf{R} / \gamma(\mathbf{R} \cdot \beta) \right]^2}} dW
\]

Note that as \( \frac{\mathbf{v}_o}{c} \to 0 \) the result reduces to the non-relativistic case.