Suppose one has six identical stationary charges, \( q \), placed on the vertices of the regular octahedron, shown above. The total electrostatic potential, \( V \), of this system would satisfy

\[
\nabla^2 V(r) = -4\pi \sum_{i=1}^{6} q \delta(r - r_i).
\]

What could one say about the symmetry properties of \( V(r) \)? As long as we perform operations on the octahedron which leave it in the same position, \( V(r) \) will remain unaltered. Now, if we place a particle of mass, \( m \), and charge, \( Q \), at an arbitrary position, \( r \), in this system and write its total classical energy, \( E \), we have

\[
E = \frac{\mathbf{p} \cdot \mathbf{p}}{2m} + QV(r).
\]

Using the Quantum Mechanical postulate that \( \mathbf{p} = \frac{\hbar}{i} \nabla \) we have the following Hamiltonian (energy) operator,

\[
\mathbf{H} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + QV(r) \right].
\]

The steady state Schrödinger equation becomes

\[
\mathbf{H} \Psi(r) = E \Psi(r)
\]

\[
\left[ -\frac{\hbar^2}{2m} \nabla^2 + QV(r) \right] \Psi(r) = E \Psi(r)
\]

In quantum mechanics, then, the goal is to solve this differential equation. This problem is analogous to what one finds for a free electron (\( Q = -e \)) in a crystalline solid.
What can one say about the solution, $\Psi(r)$, to this not so simple differential equation?

\[
[\nabla^2 + aV(r)]\Psi(r) = A\Psi(r)
\]

\[
L\Psi(r) = A\Psi(r)
\]

1. First, we know that $\nabla^2$ is spherically symmetric and invariant under all rotations.
2. The $V(r)$ is invariant under all the operations, $O_k$, which leave the octahedral charge distribution invariant.
3. Thus $L$ is invariant under all operations of the "group of covering operations" on the octahedron.

Using this information we can write $O_kL = LO_k$ (the $L$ is not altered by $O_k$) and

\[
O_kL\Psi(r) = O_kA\Psi(r)
\]

\[
L[O_k\Psi(r)] = A[O_k\Psi(r)].
\]

The above expression tells us that all functions $\Psi_k(r) = O_k\Psi(r)$ are also solutions to the eigenvalue equation. By using a group theory analysis, one can simplify the process of finding the solution to this not so simple differential equation. For example, from what one knows about the group of operations which leave the octahedron invariant one can determine the allowed degeneracies of the eigenstates, $\Psi_A(r)$, and their angular dependence. In order to take full advantage of these group theory "tools" one needs some basic theory.

<table>
<thead>
<tr>
<th>$O(432)$</th>
<th>$E$</th>
<th>$8C_3$</th>
<th>$3C_2$ = $3C_4^2$</th>
<th>$6C_2$</th>
<th>$6C_4$</th>
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<tbody>
<tr>
<td>$A_1$</td>
<td>1</td>
<td>1</td>
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<tr>
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<td>$E$</td>
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<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$(x^2 - y^2, 3z^2 - r^2)$</td>
<td>$(R_x, R_y, R_z)$</td>
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<tr>
<td>$T_1$</td>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
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<tr>
<td>$T_2$</td>
<td>3</td>
<td>0</td>
<td>-1</td>
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<td>-1</td>
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</table>

The table above is called the character table for the group, $O$. From this table one can conclude that the differential equation has eigenstate solutions with degeneracies 1, 2, and 3. Singly degenerate states depend only on $\rho(r)$, doubly degenerate solutions have angular dependence which is $\frac{1}{r^2} (x^2 - y^3, 3z^2 - r^2)$ and triply degenerate states have angular dependence given by the functions in the bottom two rows of the first column.
GROUP, $\mathcal{G}$, $\ast$ $=$ $A_1, A_2, A_3, A_4, \ldots$, $A_k$

where the $A_i$ are a set of elements with a group operation defined

(group multiplication) which associates a third member of the group, say $A_k$, with some other two elements $A_j$ and $A_l$

i.e. $A_j A_k = A_l$

(symbolically)

↑

group operation

(1) closure is satisfied: for all $A_j, A_k \in \mathcal{G}$, $B \in \mathcal{G}$, $B \ast A_j \ast A_k

(2) the associative law holds: $A_j (A_k A_l) = (A_j A_k) A_l$

(3) the unit element exists $\in \mathcal{G}$, $E \ast A_j = A_j = A_j \ast E$ for all $A_j \in \mathcal{G}$

(4) Every element has an inverse, $A_j^{-1}$, $A_j A_j^{-1} = A_j^{-1} A_j = E$

ORDER OF $\mathcal{G}$ = $k$ = number of elements

FINITE GROUP: has finite $k$

INFINITE GROUP: $k \rightarrow \infty$

ABELIAN GROUP: $A_j A_i = A_i A_j$ for all $i, j$ (all elements commute)

PERIOD OF $A_j$ (or ORDER OF $A_j$): $n$ such that $A_j^n = E$

CYCLIC GROUP of order $n$:

a set of $n$ elements all formed from products of $A_j$

$A_j, A_j^2, A_j^3, \ldots, A_j^n = E$

EXAMPLES OF GROUPS

(1) $\mathcal{G}$ = set of all positive and negative integers (and zero)

(group multiplication = addition)

$h = \infty$

(2) $\mathcal{G}$ = set of all $n \times n$ matrices with determinant $\neq 0$

(group multiplication = matrix multiplication)

$h = \infty$ (allowing $n \rightarrow \infty$)

(3) $\mathcal{G}$ = set of all covering operations on a circle:

$A_j = \text{all rotations about axis (through center and 1 plane)}$

(group multiplication = addition of angles)

$h = \infty$
(4) \( \mathcal{O} = \) set of all covering operations on a rectangle \( \cong D_2 \)

- \( A_1 = \) rotation by \( \pi \) about axis thru \( o \)
- \( A_2 = \) rotation by \( \pi \) thru \( 2 \)
- \( A_3 = \) " thru \( 1 \)
- \( A_4 = E \) (rotation about \( o \) by \( 2\pi \)) = \( A_1 A_1 \)

\( n = 4 \)

(5) \( \mathcal{O} = \) set of covering operations on an equilateral triangle \( \cong D_3 \)

- \( A_1 = \) rotation by \( \pi \) about \( \alpha \)-axis \( \cong A \)
- \( A_2 = \) rotation by \( \pi \) about \( \beta \)-axis \( \cong B \)
- \( A_3 = \) rotation by \( \pi \) about \( \gamma \)-axis \( \cong C \)
- \( A_4 = \) rotation by \( \frac{2\pi}{3} \) about \( \alpha \)-axis \( \cong D \) (clockwise)
- \( A_5 = \) rotation by \( -\frac{2\pi}{3} \) about \( \alpha \)-axis \( \cong F \)
- \( A_6 = E \)

\( E : \begin{array}{ccc} 3 \end{array} \)
\( A : \begin{array}{ccc} 3 \end{array} \)
\( B : \begin{array}{ccc} 2 \end{array} \)
\( C : \begin{array}{ccc} 1 \end{array} \)
\( D : \begin{array}{ccc} 3 \end{array} \)
\( F : \begin{array}{ccc} 2 \end{array} \)

- \( EA : \begin{array}{ccc} 1 \end{array} \)
- \( EA = F \)
- \( CA : \begin{array}{ccc} 2 \end{array} \)
- \( CA = D \)
- \( DA : \begin{array}{ccc} 3 \end{array} \)
- \( DA = C \)
- \( FA : \begin{array}{ccc} 1 \end{array} \)
- \( FA = B \)
- \( AB : \begin{array}{ccc} 2 \end{array} \)
- \( AB = D \neq BA \)
- \( BB : \begin{array}{ccc} 2 \end{array} \)
- \( BB = E \) (\( B \) has order 2)
- \( DD : \begin{array}{ccc} 3 \end{array} \)
- \( DD = F \)
- \( FF : \begin{array}{ccc} 3 \end{array} \)
- \( FF = D \)

\( FDD = D^3 = E \) (\( D \) has order 3)
\( FFF = F^3 = E \) (\( F \) has order 3)

\( A^{-1} = A \)
\( B^{-1} = B \)
\( C^{-1} = C \)
\( D^{-1} = F \)
\( F^{-1} = D \)
GROUP MULTIPLICATION TABLE

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GROUP of order 6

PROPERTIES OF THE GROUP MULTIPLICATION TABLE:

(i) A given element appears in a given row (or column) once and only once. This is called the "Rearrangement Theorem."

Proof: (For rows; similar argument holds for columns)

(a) Consider the row with (first element) \(A_r\). This row consists of
\[A_r, A_r A_2, A_r A_3, \ldots, A_r A_{h-1}, A_r A_h\]
and there are only \(h\) elements in the row.

(b) Let \(A_i \neq A_k\) and assume that \(A_r A_i = A_r A_k = A_l\)
but this \(\Rightarrow A_i = A_k A_k^{-1}\) and \(A_k = A_i A_i^{-1}\) and \(A_k = A_i\).
This is a contradiction: for \(A_i \neq A_k\), \(A_r A_i \neq A_r A_k\) and an element appears only once.

(c) Since there are \(h\) elements in a row and they are all different,
every element of the group appears once—and only once.

(ii) Every row (or column) is different from every other row (or column).

(a) Consider two rows:
\[\{A_r, A_r A_2, A_r A_3, \ldots, A_r A_{h-1}\}\]
where
\[\{A_2, A_4, \ldots, A_{h-2}, A_{h-1}\}\]

(b) Assume just one corresponding product gives the same element:
i.e., \(A_r A_k = A_2 A_k = A_3\)

then \(A_r = A_2 A_k^{-1}\) and \(A_2 = A_3 A_k^{-1}\), or \(A_r = A_3\). Contradiction!

(c) No two rows have corresponding elements equal and no two columns have any two corresponding elements equal.

(3) If the group is abelian then the table is symmetric across the diagonal.
A set of elements $\mathcal{G} = \{ S_1, S_2, S_3, \ldots, S_g \}$ is a subgroup (with order $g$) of the group $\mathcal{G}$ (with order $n$) if:

1. All $S_i \in \mathcal{G}$
2. If $S_i, S_j, \ldots$ form a group under the same group operation as defined for $\mathcal{G}$
   i.e. $\forall S_i, S_j, S_i S_j \in \mathcal{G}$
   $\forall S_i, S_i^{-1} : S_i S_i^{-1} = S_i^{-1} S_i$

**Cosets:**

Let $\mathcal{G}$ be a subgroup of $\mathcal{G}$ with order $g$ and $x \in \mathcal{G}$; $x$ not in $\mathcal{G}$

$xe, xS_2, xS_3, xS_4, \ldots, xS_g \equiv$ a left coset of $\mathcal{G}$ w.r.t. $x \equiv x\mathcal{G}$

$exe, S_2 x, S_3 x, S_4 x, \ldots, S_g x \equiv$ a right coset of $\mathcal{G}$ w.r.t. $x \equiv \mathcal{G}x$

**Properties of Cosets**

1. **Cosets are not subgroups** (because they don't contain $E$)
2. $\mathcal{G} \times x$ contains no element in $\mathcal{G}$
   $x\mathcal{G}$ contains no element in $\mathcal{G}$
3. Either two right cosets are identical or they have no elements in common.
   Either two left cosets are identical.
4. Cosets contain $g$ elements (rearrangement theorem on $\mathcal{G}$)

**Proofs:**

1. **Suppose $S_i x = S_k x \in \mathcal{G}$** then $x = S_i^{-1} S_k = \text{same} S_e \in \mathcal{G}$ but then $x \in \mathcal{G}$ and $S_i x, S_2 x, S_3 x, \ldots, S_g x \equiv \mathcal{G}$ and $\mathcal{G}x$ is not a coset (rearrangement theorem)

2. **Suppose $x S_i = y S_j$ for $x$ and $y$ not in $\mathcal{G}$**
   then $y^{-1} x = S_j^{-1} S_i \in \mathcal{G}$
   or $y^{-1} x = S_k$
   but then $y^{-1} x \mathcal{G} = S_k \mathcal{G} = \mathcal{G}$ by rearrangement theorem.
   $\therefore x \mathcal{G} = y \mathcal{G}$.

**Property (3) says that two cosets (two left cosets or two right cosets) never partially overlap.**
THEOREM: In a group of order \( n \) the only possible subgroups have order \( \frac{n}{q} \) satisfying \( \frac{n}{q} = 1, 2, 3, 4, \ldots \)
\[ n/q = I = \text{index of subgroup } \delta \text{ in } G. \]

proof:

1. There are \( n \) elements, \( A_1, A_2, A_3, \ldots A_n \) in \( G \).
   Each \( A_i \) either is in \( G \) or in a coset of \( \delta \). Let \( \delta A_i \) be non-identical cosets.
   Since \( \delta \) is finite there will be a finite number, \( 2 \), of these non-identical cosets.

2. \[ G = \delta + \delta A_1 + \delta A_2 + \delta A_3 + \delta A_4 + \ldots \delta A_2 \]

3. Each coset \( \delta A_i \) is distinct and has \( g \) elements
   \[ n = g + g + g + g + \ldots g = lg \]
   and \( \frac{n}{g} = I \)

IMPROPER SUBGROUPS: \( E \) and \( G \)

PROPER SUBGROUPS: all others

ISOMORPHIC GROUPS:

1. The set of isomorphic groups with \( n = 6 \):
   - The group of permutations of 3 distinguishable objects:
     \[ \begin{align*}
     (123), (132), (123), (132), (123), (123) \end{align*} \]
   - Replace 1 with 6, replace 2 with 3, and replace 3 with 2.
   - The group operation \( AB \) is \((123)(132) = (132) = 1 \)

2. The group of covering operations on an equilateral triangle (p.6) \( \equiv D_3 \)

3. The set of 2x2 matrices whose det = 1:
   \[ \begin{align*}
   (1 \ 0), (0 \ 1), (-\frac{1}{2} \ \frac{\sqrt{3}}{2}), (\frac{\sqrt{3}}{2} \ \frac{1}{2}), (-\frac{1}{2} \ -\frac{\sqrt{3}}{2}), (-\frac{\sqrt{3}}{2} \ -\frac{1}{2})
   \end{align*} \]
   - Group operation: matrix multiplication
   \[ AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = 1 \]
C₆ is not isomorphic to the groups listed in example:

C₆ — the group of rotations by 60°, 120°, 180°, 240°, 300°, 360° about the same axis — is a group of order 6.

\[ R(60°) = A, \ R(120°) = A², \ R(180°) = A³, \ R(240°) = A⁴, \ R(300°) = A⁵, \ R(360°) = A⁶ \]

A B C D E F

This is the cyclic group of order 6. Its multiplication table is

<table>
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<th></th>
<th>A</th>
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</table>

Note: Each row and column is cyclic perm. of ABCDEF

Note: The table is symmetric w.r.t. diagonal

The following group is isomorphic to C₆:

\{ 1, 2, 3, 4, 5, 6 \} multiplication modulo 7 = group operation
\{ E, [A, [A, D], F, C \} Fill in A, B, C, D, F so that this fits table above.

\[
\begin{align*}
1 \times 2 &= 2, & 1 \times 3 &= 3 & \text{etc.} & & \ldots & & 1 \times E &= 2, \\
2 \times 2 &= 6 & & 3 \times 3 &= 9 = 2 + 7 &= 2 & & 4 \times 4 &= 16 = 2 + 14 &= 2 \\
2 \times 4 &= 1 + 7 &= 1 & & 3 \times 4 &= 12 = 5 + 7 &= 5 & & 4 \times 5 &= 20 = 6 + 14 &= 6 \\
2 \times 5 &= 3 + 7 &= 3 & & 3 \times 5 &= 15 = 1 + 14 &= 1 & & 4 \times 6 &= 24 = 3 + 21 &= 3 \\
2 \times 6 &= 5 + 7 &= 5 & & 3 \times 6 &= 18 = 4 + 14 &= 4 & & 6 \times 6 &= 36 = 1 + 35 &= 1
\end{align*}
\]

By the \( \frac{n}{q} = 2 \) theorem (p.29) a group of order 6 can have subgroups (proper subgroups) of order 2 or 3.

D₃:

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<th></th>
<th>E</th>
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(Covering op. on Δ) has subgroups:

\{A, E\}, \{B, E\}, \{C, E\} \quad q = 2

\{E, D, F\} (cyclic) \quad q = 3

C₆ has a cyclic subgroup of order 3: \{B, B² = D, B³ = E\}

others?
**Inverse of Products**

1. Let $E, A, B, C, \ldots$ be a group $G$.
2. $(AB)^{-1} (AB) = E$ since $AB$ is some element of $G$ and all elements have inverses.
3. $(AB)^{-1}(AB) = [(AB)^{-1} A] B = E$ (associative property)
   
   $$(AB)^{-1} A = EE^{-1} = B^{-1}$$
   
   definition of $E = BB^{-1}$ and $EB^{-1} = B^{-1}$

   $$(AB)^{-1} = B^{-1} A^{-1}$$

   $$\therefore (AB)^{-1} \times E$$

   $$(ABCDF)^{-1} = F^{-1} D^{-1} C^{-1} B^{-1} A^{-1}$$

**Conjugate Elements in a group, $G$:**

- $B$ is conjugate to $A$ if $\exists x \in G \forall B = x A x^{-1}$

1. **Theorem:** If $B$ is conjugate to $A$, then $A$ is conjugate to $B$.
   
   $$(B = x A x^{-1} \Rightarrow x^{-1} B x = A \quad x^{-1} \text{ is in } G \text{ and } x \text{ is inverse of } x^{-1})$$

2. **Theorem:** If $B$ and $C$ are conjugate to $A$, then $B$ and $C$ are conjugate.
   
   $$(B = x A x^{-1} \quad C = y A y^{-1} \Rightarrow A = x^{-1} B x = y^{-1} C y \Rightarrow B = (xy^{-1}) C (xy^{-1})^{-1})$$

**A Class, $C$ is a set of elements in $G$ which are mutually conjugate.**

$X \in C$ means:

- if $(k_1, k_2, k_3, \ldots, k_n) \in C$
- then $X k_i X^{-1} = \text{some } k_j$ for every $k_i$ in $C$

**Example:** Class (including $B$) in $G$:

$$C = \{E, B E^{-1}, A_2 B A_2^{-1}, A_3 B A_3^{-1}, \ldots, A_n B A_n^{-1}\}$$

if we eliminate all duplicated elements this set = Class including $B$.

**Mathematical Definition of Class, $C$, of group $G$:**

$$C = \text{set of elements of } G \forall X \in C \quad X C X^{-1} = C \quad \forall X \in G.$$
(1) E forms a class. E occurs in no other class in $\mathcal{G}_1$. $(EEE^{-1}, A_iA_i^{-1}, \ldots) = (E)\mathcal{G}_1$ (no duplicates counted)

(2) If $\mathcal{G}_1$ is Abelian then there are $n$ classes— one for each element of $\mathcal{G}_1$.

(3) Each class is distinct. If $A \in C_1$ and $A \in C_2$ then all elements of $C_1$ are conjugate to $C_2$ and $C_1 = C_2$.

(4) $\mathcal{G}_1 = E + C_1 + C_2 + C_3 + \cdots$ ( $\mathcal{G}_1$ can be decomposed into distinct classes)

(5) $E$ is the only class of $\mathcal{G}_1$ which is a subgroup.

(6) If the group elements are represented by matrices the trace of all elements in a class are equal:

$$\text{trace} \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} = \sum_i A_{ii} = A_{11} + A_{22} + A_{33} + \cdots + A_{nn}$$

(7) Note that $XBX^{-1}$ is a similarity transformation on $B$.

A class is a part of $\mathcal{G}_1$ which one can never "get out of" by any similarity transformation.

(8) Classes are sets of elements which in general represent operations of a "similar physical sort.

For example,

$$B = X^{-1}AX$$

means: first, perform $X$. Then operate with $A$. Finally transform back with $X^{-1}$. The result is the same as if you had operated only with $B$.

$x^{-1}Ax$ has the same effect as $B$.

(9) Usually physical-symmetry considerations can give the classes of a group.
EXAMPLE: CLASSES OF $D_3$

$D_3 = (A, B, C, D, F, E)$ \quad $A^2 = A, \quad B^2 = B, \quad C^3 = C, \quad D^2 = F, \quad F^2 = D$

$C_1 = \text{class including } E = E$

$C_2 = \text{class including } A:\begin{cases} EA^{-1} = A \\ A A A^{-1} = A \\ B A B^{-1} = B A B = B D = C \\ C A C^{-1} = C A C = C F = B \\ D A D^{-1} = D A F = D C = B \\ F A F^{-1} = F A D = F B = C \end{cases} = (A, B, C) = \text{all the rotations about axes in the plane of } \Delta$

(these rotations are sometimes called improper rotations)

$C_3 = \text{class including } B = \text{class including } C = (A, B, C)$

$C_4 = \text{class including } D:\begin{cases} E D E^{-1} = D \\ A D A^{-1} = F \\ B D B^{-1} = F \\ C D C^{-1} = F \\ D D D^{-1} = D \\ F D F^{-1} = D \end{cases} = (D, F) = \text{all rotations about an axis perpendicular to the plane of the equilateral } \Delta,$

(called proper rotations)

$D_3 = E + (A, B, C) + (D, F)$

classes in the permutation group of 3 distinguishable objects: $E \cong T_3$

$C_1 = E \quad \text{(no permutation)}$

$C_2 = (A, B, C) \quad \Rightarrow \text{odd number of interchanges of the 3 objects}$

$C_3 = (D, F) \quad \Rightarrow \text{even number of interchanges of the 3 objects}$

classes in $C_2$

$C_1 = E$

$C_2 = (B) \quad C_5 = (D) \quad A B A^{-1} = A B F = A A = B$

$C_3 = (A) \quad C_6 = (F) \quad C B C^{-1} = C B C = C F = B$

$C_4 = (C)$

$C_6$ is an Abelian group. Thus all the classes consist of one element.
INVARlANT SUBGROUP is a subgroup consisting of complete classes

Comments:

1. An invariant subgroup is sometimes called a "normal divisor".

2. If \( \mathcal{H} \) is invariant then \( x\mathcal{H} = \mathcal{H}x \) for all \( x \) in \( \mathcal{H} \)
   i.e., all left and right cosets (of \( \mathcal{H} \) w.r.t \( x \)) are identical

   "proof":
   
   (a) \( \mathcal{H} = C_1 + C_2 + C_3 + \ldots \) i.e., \( \mathcal{H} \) can be written as the
       sum of complete classes
   
   (b) \[ x\mathcal{H}x^{-1} = xC_1x^{-1} + xC_2x^{-1} + xC_3x^{-1} + \ldots \]
       \[ = C_1 + C_2 + C_3 + \ldots \]
       \[ = \mathcal{H} \]
   
   (c) \[ x\mathcal{H} = \mathcal{H}x \quad \text{or} \]

3. There are a finite number, \((l-1)\), of distinct cosets of \( \mathcal{H} \)
   
   \[ h = \text{order of } \mathcal{H} \]
   
   \[ g = \text{order of } \mathcal{H} \text{ (subgroup of } \mathcal{H}) \]
   
   \[ l = \text{index of } \mathcal{H} \quad (\frac{h}{g} = l) \]

   Recall that \( \mathcal{H} = \mathcal{H} + \mathcal{H}A_1 + \mathcal{H}A_2 + \ldots + \mathcal{H}A_l \)

   where \( \mathcal{H}A_i \) are distinct cosets of \( \mathcal{H} \) in \( \mathcal{H} \). Since there are \( h \) elements
   in \( \mathcal{H} \) and each coset \( \mathcal{H}A_i \) has \( g \) elements
   
   \[ h = g + (l-1)g = lg \]

   \( \Rightarrow \) There are \((l-1)\) distinct cosets of \( \mathcal{H} \) in \( \mathcal{H} \).

Example: In \( D_3 \): \( (E, D, F) = C_1 + C_3 \) is invariant subgroup. (see p I-10)

   \[ C_1 = E \quad C_3 = (D, F) \]

   \( (E, D, F) \) = the group of proper rotations (about axis \( \perp \) plane of \( \Delta \)) \( \equiv C_3 \)

   is an invariant subgroup of \( D_3 \)

Example: In \( C_6 \): \{E, B^2, D, B^2E^3 \} is a subgroup. \( = C_3 \)

It is an invariant subgroup since \( (E), (D) \) and \( (E) \) are each complete classes.
There is no similarity transformation which when operating on an element of an invariant subgroup, \( S \), can change the element into an element which is not in \( S \).

This follows from the definition of an invariant subgroup—i.e., that it is made up of complete classes.

For all \( x \), \( x S_i x^{-1} \in S \), \( S = \{ E, S_2, S_3, \ldots, S_g \} \).

**Complex** \( \mathcal{K} \) is a collection of distinct elements—disregarding arrangement \( \equiv \mathcal{K} \):

\[
\mathcal{K} \equiv (k_1, k_2, k_3 \ldots \ldots k_m) \quad (\text{has } m \text{ elements}) \quad (\text{all are different})
\]

\[
= (k_6, k_{10}, k_1, k_{m-1}, k_2 \ldots \ldots k_3)
\]

Order is not important; \( k_i \in S_i \).

**Multiplication of** \( \mathcal{K} \) **by an element of** \( S \):

\[
\mathcal{K}x = (k_1x, k_2x, k_3x, \ldots k_mx)
\]

**Eliminate any duplication**

\[
= (k_6x, k_{10}x, k_1x, k_{m}x \ldots k_3x)
\]

**Multiplication of two complexes**:

\[
\mathcal{K} \mathcal{L} = (k_1R_1, k_2R_1, k_3R_1, \ldots, k_mR_1, k_1R_2, k_2R_2, \ldots, k_mR_m)
\]

= collection of all possible products \( k_iR_j \)

The arrangement of these products is not important; but a product involving some element of the group, \( H \), is found only once in the complex no matter how many times it is generated by \( k_iR_j \).

In "complex" notation:

\[
S \mathcal{S} = S \]

\[
\times S_{\text{INV}}^{-1} = S_{\text{INV}} \quad \forall x \in S
\]
We shall now use the notation that $\mathbf{K}_i$ represents an element of a smaller group. $\mathbf{K}_i$ is a collection of elements of $\mathbf{G}$ but as a symbol represents one element of new group.

**Factor Group of $\mathbf{G}$ w.r.t. the invariant subgroup $\mathbf{H}$**

$\mathbf{F} = (\mathbf{H}, \mathbf{H}_2, \mathbf{H}_3, \mathbf{H}_4, \ldots, \mathbf{H}_n)$

$\mathbf{H}_i = \{ \mathbf{k}; \mathbf{H}_i \mathbf{H} = \text{left coset of } \mathbf{H} \text{ w.r.t } \mathbf{H}_i \text{ and } \mathbf{k} \not\in \mathbf{H} \}$

Each $\mathbf{H}_i \equiv \mathbf{k}; \mathbf{H}_i \mathbf{H}$ is distinct (appears only once in $\mathbf{F}$).

**Note:** $\mathbf{K}_i$ = collection of elements; $\mathbf{k}_i = \{\text{one element of } \mathbf{K} \not\in \mathbf{H} \}$ and $\mathbf{k}_i \neq k_\mathbf{j}$

**Comments:**

1. There are $(l-1)$ distinct $\mathbf{H}_i$. There is a 1-1 correspondence between the distinct left cosets of $\mathbf{H}$ and the $\mathbf{K}_i$. (Recall (p. 11-14) that $\mathbf{F}$ has $(l-1)$ distinct left cosets of $\mathbf{H}$ and $(l-1)$ distinct right cosets of $\mathbf{H}$.)

2. Since $\mathbf{H}$ is an invariant subgroup of $\mathbf{G}$, $\mathbf{k}_i \mathbf{H} = \mathbf{H}_i \mathbf{k}_i = \mathbf{H}_i$.

3. Group multiplication in $\mathbf{F}$ is defined to be the same as complex multiplication.

$\mathbf{F}$ is a group of order $l$ with group multiplication = complex multiplication.

1. **Closure:**
   - (a) $\mathbf{S} \mathbf{H}_i = \mathbf{S} (\mathbf{k}_i \mathbf{H}) = \mathbf{S} (\mathbf{k}_i \mathbf{H}_i) = (\mathbf{S} \mathbf{k}_i) \mathbf{H}_i = \mathbf{H}_i \mathbf{k}_i = \mathbf{H}_i$.
   - $\mathbf{S} \mathbf{K}_i = \mathbf{K}_i$ (complex: element appears only once, arrangement not important.)

2. **Identity Element:** $\mathbf{S}$.
   - $\mathbf{H}_i \mathbf{S} = \mathbf{K}_i$; $\mathbf{S} \mathbf{K}_i = \mathbf{K}_i$; $\mathbf{S} \mathbf{S} = \mathbf{S}$

3. **Inverses:** $\mathbf{k}_i = \mathbf{K}_i^{-1}$.
The subgroups of $D_3$ are:

- $(E, A) = \bar{S}_1$
- $(E, B) = \bar{S}_2$
- $(E, C) = \bar{S}_3$
- $(E, D, F) = \bar{S}_4$

The classes of $D_3$ are:

- $(A, B, C) = E$
- $(E) = E$
- $(D, F) = E$

The only invariant subgroup (consisting of complete classes) is $\bar{S}_4 = (E, D, F)$ (E is also invariant - but it is an improper subgroup.)

Cosets of $\bar{S}_4$:

- $(AE, AB, AF) = (A, B, C) = E$
- $(BE, BO, BF) = (B, C, A) = E$
- $(CE, CO, CF) = (C, A, B) = E$

:. for $D_3$:

$\bar{S}_4 = (\bar{S}_4, E)$ = factor group of $\bar{S}_4$ in $D_3$

Factor Group Multiplication Table for $D_3$

<table>
<thead>
<tr>
<th></th>
<th>$\bar{S}_4$</th>
<th>$E_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{S}_4$</td>
<td>$\bar{S}_4$</td>
<td>$E_2$</td>
</tr>
<tr>
<td>$E_2$</td>
<td>$E_2$</td>
<td>$(A, B, C)$ $(A, B, C)$</td>
</tr>
</tbody>
</table>

Complex multiplication:

$E_2 E_2 = (A, B, C)(A, B, C)$

$= (AA, AB, AC, BA, BB, BC, CA, CB, CC)$

$= (E, D, F, F, E, D, D, F, E)$

$= \bar{S}_4$

Isomorphism

A 1-1 correspondence between elements of two groups

Homomorphism

A "many to one" correspondence between elements of two groups.
THE HOMOMORPHISM BETWEEN $\mathcal{G}$ and one of its FACTOR GROUPS.

Order of $\mathcal{G} = h$  
Order of $\mathcal{H} = q$  
$\mathcal{H}$ is an invariant subgroup  
$\mathcal{K}_i$ are cosets of $\mathcal{H}$  
Order of $\mathcal{K}_i = q$  
each $\mathcal{K}_i$ contains $q$ elements of $\mathcal{G}$.

This is a 1-to-1 correspondence between $\mathcal{G}$ and $\mathcal{H}$.

CLASS MULTIPLICATION = complex multiplication counting the number of times an element appears in the product.

**Example:** in $\mathbb{D}_3$

$E_1 = E$  
$E_2 = (A, B, C)$  
$E_3 = (D, F)$

$E, E = E = E$,

$E, E_2 = (EA, EB, EC) = E_2 = E_2 E$,

$E_2 E_3 = (AD, AF, BD, BF, CD, CF) = (E, A, C, A, A, B) = (A, B, C) + (A, B, C)$

$= 2 E_2$

$E_2 E_2 = (A, B, C) (A, B, C) = (AA, AB, AC, BA, BB, BC, CA, CB, CC)$

$= (E, D, F, E, D, D, F, F) = 3 (E) + 3 (D, F) = 3 E_1 + 3 E_3$

$E, E_3 = (E) (D, F) = (D, F) = E_3 = C_3 E$,

$E_3 E_2 = (D, F) (A, B, C) = (DA, DB, DC, FA, FB, FC) = (C, A, B, B, C, A) = 2 E_2$

$E_3 E_3 = (D, F) (D, F) = (DD, DF, FD, FF) = (F, E, E, D) = 2 E_1 + E_3$