Physics 357 Final
May 9, 2005

Sample 357
Final Exam
May 2005

PUT YOUR ANSWERS ON THE SHEET PROVIDED. USE SCRATCH PAPER FOR WORK.
See attached sheets for more information.

\[ \sin^2 \theta_w = 0.23 \; ; \quad Y_R = 2Y_L = -2 \; ; \quad P_R = \frac{1}{2} [1 + \gamma^5] \]
\[ \cos \theta_w = \frac{e}{g_1} \; ; \quad \sin \theta_w = \frac{e}{g_2} \; ; \quad P_L = \frac{1}{2} [1 - \gamma^5] \]
\[ (\gamma^\mu p_\mu - m) u = 0 \quad \; ; \quad \psi_L = \psi P_R \]
\[ \hat{\rho} \cdot \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} \quad T^3 = \frac{1}{2} \quad \text{for } f_L ; \quad T^3 = 0 \quad \text{for } f_R ; \quad (f = \text{quark or lepton}) \]
\[ Q_f = \text{charge/e} \]
\[ x^\mu = (ct; r); \quad \partial \mu = \partial / \partial x^\mu = (\partial, \nabla); \]

\[ p_\mu = (E/c; p) \]

\[ \mathcal{L} = -\frac{1}{4} F^{\alpha \beta} F_{\alpha \beta} - J_\mu A^\mu; \quad F^{\alpha \beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha; \]

\[ \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^2 \phi; \quad \text{iq}[\phi^* \partial_\mu \phi - \partial_\mu \phi^*]; \]

\[ \partial_\mu \partial^\mu \phi + m^2 \phi = 0 \]

\[ \partial \mathcal{L} / \partial \phi - \partial_\mu [\partial \mathcal{L} / (\nabla_\mu \phi)] = 0 \]

in the following \( k \cdot x = k_\mu x^\mu \) and \( p \cdot x = p_\mu x^\mu \) and \( k^\mu = (k^0; k) \)

\[ \varphi(x^\mu) = \int \left[ a_k e^{ik \cdot x} + a^*_k e^{-ik \cdot x} \right] d^3k / [(2\pi)^{3/2}k^0]^{1/2}; \]

\[ A^\mu(x^a) = \int \Sigma_e [\varepsilon^\mu a(k,\pm) e^{ik \cdot x} + \varepsilon^\mu a(k,\pm)^* e^{-ik \cdot x}] d^3k / [(2\pi)^{3/2}k^0]^{1/2}; \]

\[ \psi(x^\mu) = \int \Sigma_{ps} [u(p,s) e^{ip \cdot x} + v(p,s) d^* e^{ip \cdot x}] d^3p / [(2\pi)^{3/2}p^0]^{1/2}; \]

\[ b = b(p, s=\pm) \quad \text{d}^+ = d^+(p, s=\pm) \]

QED: ..................................................................................................................................................
\[
U(1) \times SU(2) \times SU(3): \quad D_\mu = \partial_\mu - i g_1 \frac{1}{2} y B_\mu - i g_2 \frac{1}{2} \tau \cdot W_\mu - i g_3 \sum_a \Lambda^a G^a_\mu
\]

\[
\begin{align*}
\psi &\rightarrow \exp[i \chi(x^\mu)] \psi \\
\psi &\rightarrow \exp[i \frac{1}{2} \varepsilon(x^\mu) \cdot \tau] \psi \\
\psi &\rightarrow \exp[i \frac{1}{2} \sum_a \alpha^a(x^\mu) \Lambda^a] \psi \\
\mathcal{L} &= \psi [i \gamma^\mu \partial_\mu - m] \psi
\end{align*}
\]

**SU(2)**

\[
\begin{align*}
\tau^1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
\tau^2 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
\tau^3 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\end{align*}
\]

\[
\exp[i \theta \tau^k/2] = 1 \cdot \cos(\theta/2) + i \tau^k \sin(\theta/2) \quad k=1,2,3 \\
[\tau^m, \tau^n] = 2i \varepsilon^{mnk} \tau_k
\]

**SU(3)**

\[
\begin{align*}
\Lambda_k &= \begin{bmatrix} \sigma_k & 0 \\ 0 & 0 \end{bmatrix} ; k=1,2,3 \\
\Lambda_4 &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
\Lambda_5 &= \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\Lambda_6 &= \begin{bmatrix} 0 & 0 \\ 0 & \sigma_1 \end{bmatrix} \\
\Lambda_7 &= \begin{bmatrix} 0 & 0 \\ 0 & \sigma_2 \end{bmatrix} \\
\Lambda_8 &= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}
\end{align*}
\]

\[
[A_a, A_b] = 2i f_{abc} \Lambda^c; \quad i=1,2,3 \Rightarrow \text{red, green, blue} \\
f_{abc} = -f_{bac} = f_{cba}; \quad f_{123} = 1; \quad f_{145} = f_{516} = f_{246} = f_{257} = f_{345} = f_{637} = \frac{1}{2}; \quad \text{others} = 0.
\]

\[
\exp[i \theta \Lambda^a/2] = (1 - \Lambda^a \Lambda_a) + \Lambda^a \Lambda_a \cos(\theta/2) + i \Lambda^a \sin(\theta/2) \quad (\text{no sum on } a)
\]

See next page for “time line” of universe since Big Bang.
Please put your answers on this sheet. No scratch sheets will be collected for this problem. p.1
I. Fill in the blanks with the best, simplest answer. Perform all calculations which are possible from the information given; it is not necessary to show work. However, some partial credit will be given for incomplete answers. The following constants and notation are used:

\[ G_N = 6.67 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2} \]
\[ \varphi = \text{complex scalar field operator} \]
\[ u(p, \pm) = \text{fermion spinor} \]
\[ L = \text{Lagrangian density; } r^2 = x^2 + y^2 + z^2 \]
\[ A_\mu = \text{photonic field operator} \]
\[ m_e = \text{electron mass} = 0.511 \text{ MeV} \]
\[ e = \text{electronic charge} \]
\[ m_p = \text{proton mass} = 938 \text{ MeV} \]
\[ u, d, c, s = \text{quarks} \]
\[ q = \text{quark} \]
\[ \psi = \text{fermion field operator} \]
\[ \text{SM = Standard Model} \]

Fill in the blanks (3 points each, except as marked).

a) The number of gauge bosons in an SU(5) invariant local gauge theory is \( n = \) \( n^2 - 1 \) where \( n = 5 \)

b) i\( \not{\psi} \gamma^\mu \not{A}_\mu \) is invariant under all Lorentz transformations \( \text{True} \) (true or false)?

c) The SU(3)_{colour} \times SU(2)_{isospin} \times U(1)_{electric charge} gauge particles are \( \text{8 quarks, 3 vector bosons, 1 photon} \) (give names) (6 points)

d) The exact SU(2) \times U(1) invariance of the Standard Model is assumed to be broken when the average kinetic energy of the existing "particles" in the universe reaches about \( M_{\text{c}} \) where \( M \) is the rest mass of the \( W^+ \), \( e^- \) particle(s).

e) The quark composition of the \( \Omega \) is \( \text{(single color, single flavor, one strange quark)} \) (include correct color combination).

f) What phenomena has been proposed to explain the smooth (homogeneous black body) and flat (having just the critical density) universe we observe from our position in the Milky Way galaxy? \( \text{inflation} \). At about what time in the development of the universe did this occur? \( 10^{-34} \) sec.

g) The "Planck" era of the universe lasts roughly up to \( 10^{-43} \) sec. after the Big Bang.

h) The \( W^+ \) and \( W^- \) bosons give rise to charged \( \text{charged, scalar, or neutral?} \) \( \text{weak current interactions} \)

\( \text{K}_\text{p}^{[(\text{2\pi}n_\Lambda)^4]} \) acts on \( u_{\text{red}}, u_{\text{green}}, u_{\text{blue}} \), the resulting triplet is \( \left[ \begin{array}{c} u_{\text{red}} \\ u_{\text{green}} \\ u_{\text{blue}} \end{array} \right] \)

i) The \( \Omega_{\text{SM}} \) for the Standard Model is invariant under (answer yes or no): rotations in flavor space \( \text{Yes} \), rotations in color space \( \text{Yes} \), transformations of quarks into leptons \( \text{No} \), transformations of fermions into bosons \( \text{No} \), changes in complex phase of the fermion field operators \( \text{Yes} \) (6 points)

j) The proton can decay in the \( SU(5) \) Grand Unified \( \text{(give name of model) non-abelian gauge field theory} \)

k) In W\( ^+ \)'s beta decay experiment detecting non-conservation of parity, the intermediate \( W^- \) particle spin is oriented \( \text{along, opposite to} \) the magnetic field direction. (4 points) \( W^- \) spin is along \( \vec{B} \), \( \vec{B} \) are spins of \( \vec{e}^- \) and \( e^- \)

l) The top quark was produced in a \( t \bar{t} \) pair. If a 175 GeV proton collides with an oppositely directed anti-proton from a storage ring, the minimal anti-proton kinetic energy to produce the top quark pair would have to be \( \text{175 GeV} \). (5 points)

m) Which gauge invariances (i.e. \( \text{SU}(n) \)) give rise to the unification of the electromagnetic and weak interactions? \( \text{SU}(2) \times U(1) \)

n) The gluon interaction with quarks becomes \( \text{weak} \) (weak, strong) at short range and \( \text{strong} \) (weak, strong) at long range.

\( A = 100 \text{ GeV} \) left-handed electron travels along the \( (x + y + z) \) direction. The spin \( \text{ref.} \) \( \frac{1}{2} \)

\[ \text{q) In QCD gauge field theory a three gluon vertex is allowed because of what property of the gluons?} \]

\( \text{quarks carry color charge} \Rightarrow \text{gluons can interact with gluons} \)

\( r) \text{ Which of the following particles (u, c, d, s, e, \bar{e}) can not undergo a flavor change} \) \( d \rightarrow c \), \( c \)?

\( s) \text{ The (untransformed) covariant derivative for the SU(2)_{isospin} \text{ invariant local gauge field theory}} \text{ is given by} \)

\[ D \mu = \partial \mu - i g_\varphi (W^\mu_\varphi / 2 + \bar{W}_\varphi W^\mu_\varphi) \]

\[ \mu = (W^\mu, W^\mu, W^\mu) \]

\( t) \text{ The Lagrangian for the free fermion (giving rise to the Dirac equation) is} \psi \left( i \gamma^\mu \partial_\mu - m \right) \psi \) \text{. Give the expression for the proper, SU(3) \text{ gauge invariant Lagrangian for the Dirac eq.} \psi \left( i \gamma^\mu \partial_\mu - m \right) \psi \)}

\( u) \text{ The universe is believed to be about} \text{13} \cdot 10^9 \text{ years old} \)

\( v) \text{ The Standard Model Lagrangian allows which of the following phenomena? (answer yes or no) (8 points total)} \)

1) \( \gamma^\mu = \gamma \) \( \text{No} \)
2) \( p = c^+ + \pi^0 \) \( \text{Yes} \)
3) \( \gamma^\mu = c^+ + c^- \) \( \text{Yes} \)
4) \( Z_0 = e^- + \mu^- \) \( \text{No} \)
5) \( e^- + \nu_e \rightarrow W^- \) \( \text{Yes} \)
6) quark confinement \( \text{Yes} \)
7) \( W^- + u^+_l = d^+_l \) \( \text{No} \)
8) \( \gamma^\mu = \gamma + \gamma \) (\text{act only on left-handed quarks}) \( \text{No} \)

(\text{Ve has no charge, photon interacts only with charged particles})
II. Using only quark, lepton and gauge boson lines draw a "schematic" Feynman diagram corresponding to the following processes as prescribed by the Standard Model Lagrangian. The information sheet might be useful. **Label all lines with particle symbols and arrows to indicate direction of motion.** Indicate any composite particles with brackets. Each diagram should contain only allowed vertices with particle lines prescribed precisely by the Standard Model Lagrangian. (40 points)

A. $e^- + p \longrightarrow n + v_e$

B. $\pi^- \longrightarrow \mu^- + \text{neutrino}$

C. $u_{\text{red}} + d_{\text{blue}} \longrightarrow u_{\text{blue}} + d_{\text{red}}$  \hfill (u and d are the up and down quarks)

D. $p + \bar{p} \longrightarrow W^- + \pi^+ + \pi^0$
III. Give descriptive answers to the following using words and any symbols on the information sheet (or symbols defined within the description). (total of 30 points)

A. Explain, using words, symbols, and equations etc. what the "Standard Model covariant derivative" is, why it is important, and how it is used to formulate the $\mathcal{L}_\text{SM}$ for the $U(1) \times SU(2)_\text{em} \times SU(3)_\text{color}$ invariant local gauge field theory. Use the Dirac Lagrangian, $\mathcal{L} = \bar{\psi} \gamma^\mu \partial_\mu \psi$, as a starting point. Write down the SM covariant derivative, and describe

\[
\mathcal{D}_\mu = \partial_\mu - i \frac{g}{2} \gamma^\nu (\mathbf{W}_\mu)_\nu^a G_a^\mu - \frac{g}{2} \sum_{a=1}^{3} \alpha^a \mathbf{G}_a^\mu
\]

When $\mathcal{D}_\mu$ is substituted in $\mathcal{L} = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi$, one obtains

\[
\mathcal{L}_\text{new} = \bar{\psi} (2 \gamma^\mu \partial_\mu - m) \psi + \mathbf{g}^a \mathbf{G}_a \cdot \bar{\psi} \gamma^\mu \psi \mathbf{B}_\mu + \frac{1}{2} \mathbf{g}^a \mathbf{G}_a^\mu \bar{\psi} \gamma^\mu \psi \mathbf{W}_\mu
\]

The resulting $\mathcal{L}_\text{new}$ is invariant under rotations in flavor space, rotations in color space and under $\psi \to \psi' = e^{-iX(x)} \psi$ (when the gauge fields are also transformed). Terms 1, 2, and 3 give rise to the interactions of the quarks and leptons with the $\mathbf{B}_\mu$, $\mathbf{W}_\mu$, and $\mathbf{G}^\mu_a$ gauge bosons. After writing $\mathbf{B}_\mu$ and $\mathbf{W}^\mu$ in terms of the $Z_\mu$ and $A_\mu$ (photon) one obtains all the interactions in the Standard model:

\[
\begin{align*}
\text{u-d} \quad \text{e}^- \quad \nu_e \quad \nu_\mu \quad \nu_\tau
\end{align*}
\]

B. What is meant by the terms "local" and "non-Abelian" in a gauge field theory? Which part of the Standard Model involves a non-Abelian gauge symmetry and what important consequence does this have for the Standard Model? Give examples of the non-Abelian gauge operators.

\[
\text{local gauge field theory} \Rightarrow \mathcal{L} \text{ is invariant under transformations of } \psi_f \text{ of the form: } \psi_f \to \psi'_f = e^{iX(x^\mu)} \psi_f \text{ where } X(x^\mu) \text{ depends on } x^\mu \text{ and } t. \text{ Each point in space and time can undergo a "local" complex phase transformation.}
\]

\[
\text{non-Abelian: } \Rightarrow \mathcal{L} \text{ is invariant under transformations on } \psi_f \text{ of the form } \psi_f \to \psi'_f = \exp \left( \frac{i}{\alpha} \mathbf{e}(\mathbf{x}) \right) \psi_f \text{ where the } \psi_f \text{ do not commute with each other: } \psi_f \mathbf{G}_a \neq \mathbf{G}_a \psi_f.
\]

This gives rise to terms in $\mathcal{L}$ for which the gauge particle can interact with each other: the gauge particles $W^\mu$ carry "flavor". (In $SU(3)$, gluons carry "colorcharge")
SOLUTIONS:

IIIc.

In the early universe for times after $10^{44}$ sec, the temperature is high and the "thermal energy" corresponds to $kT = 10^9$ GeV. In this era all the gauge bosons (the $X$, $Y$, the gluons, the $W^+$, $W^-$, the $Z^0$ and the photon) of the SU(5) symmetric Grand Unified Theory (GUT) are MASSLESS. This massless property means that the forces the bosons mediate in exchange processes have the same strength and (infinite) range. As such the 24 gauge bosons of SU(5) represent a UNIFIED or "one-force" field.

As the universe expands and cools to temperatures characteristic of thermal energies ($kT$) equal to about $10^{10}$ GeV, a phase transition called a "spontaneous symmetry breaking" occurs. In this phase transition, the background Higgs field (corresponding to a scalar charged particle) takes on a new potential energy state, $H$. This shift in the Higgs field adds a term proportional to $H$ to that part of the Lagrangian expressing the interaction between the Higgs particle and the gauge bosons. The additional (constant) $H$ term acts like a "mass term" for the gauge boson and, in effect produces a universe in which the $X$ and the $Y$ gauge bosons have mass. This breaks the SU(5) symmetry and the GUT "force fields" are no longer unified. The forces due to the $X$ and the $Y$ become very short range (due to the large $X$ and $Y$ masses which are $\approx 10^{16}$ GeV). At this point only the SU(2)$_{bos}$ x U(1) of the Standard Model (SM) remain unified. The SU(3)$_{bos}$ gauge boson field also remains separately unified (the eight gluons represent a "unified field").

During the subsequent time period (before $kT$ drops to about 100 GeV), the Standard Model SU(2)$_{bos}$ x U(1) and the SU(3)$_{bos}$ symmetries are valid. The corresponding gauge bosons (the $W^+$, $W^-$, the $Z^0$ and the photon) remain MASSLESS and "unified" into one field with the same range and strength. However near $kT \approx 100$ GeV, a second Higgs spontaneous symmetry breaking occurs (with a second Higgs background field) and the $W^+$, $W^-$, the $Z^0$ gauge bosons take on mass close to this thermal energy, $\approx 100$ GeV. When this second phase transition occurs the SU(2)$_{bos}$ x U(1) symmetry is broken and the range of the "weak" flavor changing and the "weak" neutral current forces become short. The photon and the gluons REMAIN MASSLESS.

Today the U(1) gauge symmetry of QED remains valid as does the (separate) SU(3)$_{bos}$ symmetry of QCD. However, properties of the SU(5) and SU(2)$_{bos}$ x U(1) (the electro-weak symmetry) can still be explored in accelerators where particles are given large kinetic energies and forced to collide with large center of mass energies. At these high energies, remnants of the (now broken) symmetries are observable. The large center of mass energies in the high energy collisions allow the particles to approach each other to within small separation distances where all the forces become large and (to first approximation) about the same magnitude. We never are able to "reunify" the fields. But we can examine properties of these once valid gauge field theories.

IIId.

The photon field (reflecting the unification of the E and B fields of E&M) remains "unified". Recall that the $A_{\mu} = (V, A)$ and together these determine the electric and magnetic "fields". U(1)$_{E&M}$ is STILL a valid symmetry of the universe and gives rise to Maxwell's equations which govern all of the interactions between charged particles.

The gluon field (reflecting the unification of the 8 color-anticolor gluon fields) remains "unified" also. The SU(3)$_{bos}$ symmetry is still an unbroken symmetry of the universe -- and gives rise to the quark confinement in protons and neutrons. This strong force has a peculiar range dependence -- as you will recall. Because the gluons carry color charge and can interact the force is weak at short range and strong at long range.