Relativistic Invariance
(Lorentz invariance)

The laws of physics are invariant under a transformation between two coordinate frames moving at constant velocity w.r.t. each other.

(The world is not invariant, but the laws of physics are!)
Review: Special Relativity

Einstein’s assumption: the speed of light is independent of the (constant) velocity, \( \mathbf{v} \), of the observer. It forms the basis for special relativity.

\[
\text{Speed of light} = C = \frac{|r_2 - r_1|}{(t_2 - t_1)} = \frac{|r_2' - r_1'|}{(t_2' - t_1')}
= \frac{|dr/dt|}{(t_2 - t_1)} = \frac{|dr'/dt'|}{(t_2' - t_1')} \]
\[ C^2 = \frac{|dr|^2}{dt^2} = \frac{|dr'|^2}{dt'^2} \]
Both measure the same speed!

This can be rewritten:

\[ d(ct)^2 - |dr|^2 = d(ct')^2 - |dr'|^2 = 0 \]

\[ d(ct)^2 - dx^2 - dy^2 - dz^2 = d(ct')^2 - dx'^2 - dy'^2 - dz'^2 \]

\[ d(ct)^2 - dx^2 - dy^2 - dz^2 \text{ is an invariant!} \]
\[ \text{It has the same value in all frames (} = 0 \text{).} \]

\[ |dr| \text{ is the distance light moves in } dt \text{ w.r.t the fixed frame.} \]
A Lorentz transformation relates position and time in the two frames. Sometimes it is called a “boost”.

Lorentz Transformation

The primed frame moves with velocity $v$ in the $x$ direction with respect to the fixed reference frame. The reference frames coincide at $t=t'=0$. The point $x'$ is moving with the primed frame.

\[ x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \]
\[ y' = y \]
\[ z' = z \]
\[ t' = t - \frac{vx}{c^2} \sqrt{1 - \frac{v^2}{c^2}} \]
How does one “derive” the transformation? Only need two special cases.

1. Let $|dr|$ be the distance light travels in $dt$

$$(cdt)^2 - (dx)^2 = (cdt')^2 - (dx')^2$$

Recall the picture of the two frames measuring the speed of the same light signal.
\[(cdt)^2 - (dx)^2 = \left[acd + bdx\right]^2 - \left[fcdt + hdx\right]^2\]
\[= a^2 (cdt)^2 + b^2 (dx)^2 + 2ab (cdt)dx - f^2 (cdt)^2 - h^2 (dx)^2 - 2fh (cdt)dx\]
\[= \left[a^2 - f^2\right] (cdt)^2 + \left[b^2 - h^2\right] (dx)^2 + [2ab - 2fh] cdt dx\]

\[2ab - 2fh = 0; \quad [a^2 - f^2] = 1; \quad [b^2 - h^2] = -1\]
You can show that the following works

\[
a = h = \gamma = \left[ 1 - \beta^2 \right]^{-1/2} \\
b = -\beta \gamma = f
\]

\[
[a^2 - f^2] = \frac{1}{1 - \beta^2} - \beta^2 \frac{1}{1 - \beta^2} = 1
\]

\[
[b^2 - h^2] = \beta^2 \frac{1}{1 - \beta^2} - \frac{1}{1 - \beta^2} = -1
\]

\[
2ab - 2fh = -2\beta \gamma^2 - \left[ -2\beta \gamma \right]^2 = 0
\]
2. What is $\beta$? We need another case. Suppose a particle is at rest in the moving frame.

$$dx' = 0 \quad \text{and} \quad dx = vdt$$

$$\begin{bmatrix} cdt' \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma & -\beta \gamma & 0 & 0 \\ -\beta \gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cdt \\ vdt \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} cdt' \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma(cdt) - \beta \gamma(vdt) \\ -\beta \gamma(cdt) + \gamma(vdt) \\ 0 \\ 0 \end{bmatrix}$$
\[ \begin{align*}
\frac{c dt'}{dt} &= c \gamma \left(1 - \beta \frac{V}{c}\right) dt \\
0 &= c \gamma \left[-\beta + \frac{V}{c}\right] \Rightarrow \beta = \frac{V}{c} \\
\frac{dt'}{dt} &= \frac{1}{\sqrt{1 - \beta^2}} [1 - \beta^2] = dt / \gamma \\
L &= \begin{bmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \Rightarrow \beta = \frac{V}{c}
\end{align*} \]
Lorentz Transformation of Four-vectors

The Lorentz-transformation of both space-time and momentum-energy four-vectors can be expressed in matrix form.

\[
\begin{bmatrix}
ct' \\
x' \\
y' \\
z'
\end{bmatrix} = \begin{bmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
ct \\
x \\
y \\
z
\end{bmatrix}
\]

\[
\begin{bmatrix}
E'/c \\
p_x' \\
p_y' \\
p_z'
\end{bmatrix} = \begin{bmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
E/c \\
p_x \\
p_y \\
p_z
\end{bmatrix}
\]

\[
\beta = \frac{v}{c}
\]

\[
\begin{bmatrix}
ct \\
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
\gamma & \beta \gamma & 0 & 0 \\
\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
ct' \\
x' \\
y' \\
z'
\end{bmatrix}
\]

\[
\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

Replacing \( v \) by \( -v \) gives the inverse of the transformation.
covariant and contravariant components*

\[
\begin{aligned}
\begin{bmatrix}
ct \\
-x \\
-y \\
-z \\
\end{bmatrix} &= \text{covariant components} \\
\begin{bmatrix}
ct \\
x \\
y \\
z \\
\end{bmatrix} &= \text{contravariant components}
\end{aligned}
\]

*For more details about contravariant and covariant components see http://web.mst.edu/~hale/courses/M402/M402_notes/M402-Chapter2/M402-Chapter2.pdf
metric tensor relates components

\[
\begin{bmatrix}
ct \\
-x \\
-y \\
-z
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
ct \\
x \\
y \\
z
\end{bmatrix}

= g
\begin{bmatrix}
ct \\
x \\
y \\
z
\end{bmatrix}
\]

\[g = \text{metric tensor}\]
Using indices instead of $x, y, z$

\[
\begin{bmatrix}
x^0 \\
-x^1 \\
-x^2 \\
-x^3
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
x^0 \\
x^1 \\
x^2 \\
x^3
\end{bmatrix}
\]

\[= g \begin{bmatrix}
x^0 \\
x^1 \\
x^2 \\
x^3
\end{bmatrix}\]

\[g = \text{metric tensor}\]
4-dimensional dot product

You can think of the 4-vector dot product as follows:

\[
\begin{bmatrix}
  ct, & -x, & -y, & -z
\end{bmatrix}
\begin{bmatrix}
  ct \\
  x \\
  y \\
  z
\end{bmatrix} = (ct)^2 - x^2 - y^2 - z^2
\]
Why all these minus signs?

- Einstein’s assumption (all frames measure the same speed of light) gives:

\[
d(ct)^2 - dx^2 - dy^2 - dz^2 = 0
\]

From this one obtains the speed of light.

It must be positive!

\[
c = [dx^2 + dy^2 + dz^2]^{1/2} / dt
\]
Four dimensional gradient operator

\[
\begin{bmatrix}
\frac{\partial}{\partial ct} \\
-\frac{\partial}{\partial x} \\
-\frac{\partial}{\partial y} \\
-\frac{\partial}{\partial z}
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial ct} \\
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{bmatrix}
\]

\[
= g
\]

\[
g = \text{metric tensor}
\]
4-dimensional vector component notation

- $x^\mu$ stands for
  
  $x^0, x^1, x^2, x^3$ for $\mu=0,1,2,3$
  
  $ct, x, y, z = (ct, r)$

- $x_\mu$ stands for
  
  $x_0, x_1, x_2, x_3$ for $\mu=0,1,2,3$
  
  $ct, -x, -y, -z = (ct, -r)$
partial derivatives

\[ \partial/\partial x^\mu \equiv \partial_\mu \quad \text{stands for} \]

\[ (\partial/\partial (ct) , \partial/\partial x , \partial/\partial y , \partial/\partial z) \]

\[ = \ (\partial/\partial (ct) , \nabla) \]
partial derivatives

\( \partial^\mu \) stands for

\[
\left( \partial/\partial (ct) \, , \, -\partial/\partial x \, , \, -\partial/\partial y \, , \, -\partial/\partial z \right)
\]

\[
= \left( \partial/\partial (ct) \, , \, -\nabla \right)
\]
Invariant dot products using 4-component notation

\[ x_\mu x^\mu = \sum_{\mu=0,1,2,3} x_\mu x^\mu \]

(repeated index \(\Rightarrow\) summation implied)

\[ = (ct)^2 -x^2 -y^2 -z^2 \]
Invariant dot products using 4-component notation

\[ \partial_\mu \partial^\mu = \sum_{\mu=0,1,2,3} \partial_\mu \partial^\mu \]

(repeated index \(\Rightarrow\) summation implied)

\[ = \frac{\partial^2}{\partial (ct)^2} - \nabla^2 \]

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]
Any four vector dot product has the same value in all frames moving with constant velocity w.r.t. each other.

Examples:

\[
\begin{align*}
\mathbf{x}_\mu \mathbf{x}^\mu & \quad \mathbf{p}_\mu \mathbf{x}^\mu \\
\mathbf{p}_\mu \mathbf{p}^\mu & \quad \partial x_\mu \partial x^\mu \\
\mathbf{p}_\mu \partial x^\mu & \quad \partial x_\mu A^\mu
\end{align*}
\]
Lorentz Invariance

- Lorentz invariance of the laws of physics is satisfied if the laws are cast in terms of four-vector dot products!
- Four vector dot products are said to be “Lorentz scalars”.
- In the relativistic field theories, we must use “Lorentz scalars” to express the interactions.