Fermions and the Dirac Equation

In 1928 Dirac proposed the following form for the electron wave equation:

\[
\begin{bmatrix}
i \gamma^\mu \partial_\mu - m \cdot 1 & \end{bmatrix}
\psi(r, t) = 0
\]

The four \( \gamma^\mu \) matrices form a Lorentz 4-vector, with components, \( \mu \). That is, they transform like a 4-vector under Lorentz transformations between moving frames. Each \( \gamma^\mu \) is a 4x4 matrix.
Pauli spinors:

\[
\gamma^0 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]

\[
\gamma^1 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & \sigma^1 \\
-\sigma^1 & 0
\end{bmatrix}
\]

\[
\gamma^2 = \begin{bmatrix}
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \\
0 & i & 0 & 0 \\
-i & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & \sigma^2 \\
-\sigma^2 & 0
\end{bmatrix}
\]

\[
\gamma^3 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & \sigma^3 \\
-\sigma^3 & 0
\end{bmatrix}
\]

\[
\gamma^{0\dagger} = \gamma^0 \\
\gamma^{k\dagger} = -\gamma^k
\]

\[
\sigma^1 = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

\[
\sigma^2 = \begin{bmatrix}
0 & -i \\
i & 0
\end{bmatrix}
\]

\[
\sigma^3 = \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]
The $0$ and $1,2,3$ matrices anti-commute

The $1,2,3$, matrices anti-commute with each other

The square of the $1,2,3$, matrices equal minus the unit matrix

The square of the $0$ matrix equals the unit matrix

All of the above can be summarized in the following expression:

$$\left[ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu \right] = 2g^{\mu \nu} \cdot 1$$

Here $g^{\mu \nu}$ is not a matrix, it is a component of the inverse metric tensor.
With the above properties for the $\gamma$ matrices one can show that if $\Psi$ satisfies the Dirac equation, it also satisfies the Klein Gordon equation. It takes some work.

Klein-Gordon equation:

$$\left[\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right] 1 \cdot \Psi = 0$$

Dirac derived the properties of the $\gamma$ matrices by requiring that the solution to the Dirac equation also be a solution to the Klein-Gordon equation. In the process it became clear that the $\gamma$ matrices had dimension 4x4 and that the $\Psi$ was a column matrix with 4 rows.
The Dirac equation in full matrix form with \( \hbar = c = 1 \):

\[
\begin{bmatrix}
\gamma^0 \\
i \\
\frac{\partial}{\partial t} + i \gamma^1 & \frac{\partial}{\partial x} + i \gamma^2 \\
\gamma^3 & -m
\end{bmatrix}
\begin{bmatrix}
\Psi_1(r,t) \\
\Psi_2(r,t) \\
\Psi_3(r,t) \\
\Psi_4(r,t)
\end{bmatrix}
= e^{-ip\mu x^\mu} \begin{bmatrix}
u_1(p, E, m) \\
u_2(p, E, m) \\
u_3(p, E, m) \\
u_4(p, E, m)
\end{bmatrix}
\]

Spin dependence:

Space-time dependence:
After taking partial derivatives ... note 2x2 blocks.
Writing the above as a 2x2 (each block of which is also 2x2)
Incorporate the p and E into the matrices ...

\[
\begin{pmatrix}
E & 0 \\
0 & -E
\end{pmatrix}
- \begin{pmatrix}
0 & \sigma^1 p_x \\
-\sigma^1 p_x & 0
\end{pmatrix}
- \begin{pmatrix}
0 & \sigma^2 p_y \\
-\sigma^2 p_y & 0
\end{pmatrix}
\]

\[
- \begin{pmatrix}
0 & \sigma^3 p_z \\
-\sigma^3 p_z & 0
\end{pmatrix}
- \begin{pmatrix}
m & 0 \\
0 & m
\end{pmatrix}
\]

\[e^{-ip_{\mu}x^{\mu}} \begin{pmatrix}
u_a \\
u_b
\end{pmatrix} = \begin{pmatrix}0 \\
0
\end{pmatrix}\]
The exponential (non-zero) can be cancelled out.

\[
\begin{bmatrix}
E & 0 \\
0 & -E
\end{bmatrix}
- \begin{bmatrix}
0 & \bar{\sigma} \cdot \bar{p} \\
-\bar{\sigma} \cdot \bar{p} & 0
\end{bmatrix}
- \begin{bmatrix}
m & 0 \\
0 & m
\end{bmatrix}
\begin{bmatrix}
u_a \\
u_b
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
E - m & 0 \\
0 & -(E - m)
\end{bmatrix}
- \begin{bmatrix}
0 & \bar{\sigma} \cdot \bar{p} \\
-\bar{\sigma} \cdot \bar{p} & 0
\end{bmatrix}
\begin{bmatrix}
u_a \\
u_b
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
(E - m) & \bar{\sigma} \cdot \bar{p} \\
\bar{\sigma} \cdot \bar{p} & -(E - m)
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
- \begin{bmatrix}
(E + m) & \bar{\sigma} \cdot \bar{p} \\
\bar{\sigma} \cdot \bar{p} & -(E + m)
\end{bmatrix}
\begin{bmatrix}
u_3 \\
u_4
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
Finally, we have the relationships between the upper and lower spinor components.
\[
\begin{aligned}
\mathbf{u}_{\text{PARTICLE} \ (E > 0)} &= \begin{bmatrix}
    u_1(p) \\
    u_2(p) \\
    u_3(p) \\
    u_4(p)
\end{bmatrix}_{\text{PARTICLE} \ (E > 0)} = C_1 \cdot \begin{bmatrix}
    u_1 \\
    u_2 \\
    \bar{\sigma} \cdot \bar{p} \\
    (E + m)
\end{bmatrix} \\
\mathbf{u}_{\text{ANTI-PARTICLE} \ (E > 0)} &= \begin{bmatrix}
    u_1(p) \\
    u_2(p) \\
    u_3(p) \\
    u_4(p)
\end{bmatrix}_{\text{ANTI-PARTICLE} \ (E < 0)} = C_2 \cdot \begin{bmatrix}
    \bar{u}_3 \\
    \bar{u}_4 \\
    \bar{\sigma} \cdot \bar{p} \\
    (E - m)
\end{bmatrix}
\end{aligned}
\]

Convention for normalization:
\[
\bar{\mathbf{u}} \mathbf{u} = \mathbf{u}^\dagger \gamma^0 \mathbf{u} = 2m
\]
Spinors for the particle with $p$ along $z$ direction

$p$ along $z$ and spin $= +1/2$

$$u^s_{p_{(\text{PARTICLE } E>0)}} = \sqrt{E + m} \begin{pmatrix} 1 \\ 0 \\ \frac{p}{E+m} \\ 0 \end{pmatrix}$$

$$\bar{u}u = 2m$$

$p$ along $z$ and spin $= -1/2$

$$u^s_{p_{(\text{PARTICLE } E>0)}} = \sqrt{E + m} \begin{pmatrix} 0 \\ 1 \\ -\frac{p}{E+m} \\ 0 \end{pmatrix}$$

$$\bar{u}u = 2m$$
Spinors for the anti-particle with $p$ along $z$ direction

$p$ along $z$ and spin = $+1/2$

$$\psi^s = +1/2_{\text{ANTI-PARTICLE } (E > 0)} = -\sqrt{E + m} \begin{pmatrix} \frac{p}{E+m} & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$p$ along $z$ and spin = $-1/2$

$$\psi^s = -1/2_{\text{PARTICLE } E > 0} = \sqrt{E + m} \begin{pmatrix} \frac{-p}{E+m} & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$
Field operator for the spin ½ fermion

\[ \Psi = \int \int \int \sum_{s=\pm 1/2} \left[ \alpha_p^{s\mu}(p)e^{-ip\cdot x^\mu} + b_p^{\dagger s}(p)e^{ip\cdot x^\mu} \right] \frac{d^3 p}{(2\pi)^3 \sqrt{2E}} \]

Spinor for antiparticle with momentum \( p \) and spin \( s \)

Creates antiparticle with momentum \( p \) and spin \( s \)

Note: \( p_\mu p^\mu = m^2 c^2 \)
The conjugate field operator, $\bar{\Psi}$, is defined as follows:

$$\bar{\Psi} = \Psi^\dagger \gamma^0$$

$$\bar{\Psi} = \int \int \int \sum_{s=\pm 1/2} \left[ a_p^s \tilde{u}^s(p)e^{ip\cdot x_\mu} + b_p^s \tilde{v}^s(p)e^{-ip\cdot x_\mu} \right] \frac{d^3p}{(2\pi)^3 \sqrt{2E}}$$

Creates particle with momentum $p$ and spin $s$

The creation and annihilation operators obey anti-commutation relations,

$$\{a_p^r, a_p^{s\dagger}\} = \delta(\vec{p} - \vec{p}') \delta^{rs}$$

$$\{b_p^r, b_p^{s\dagger}\} = \delta(\vec{p} - \vec{p}') \delta^{rs}$$

where the anti-commutator is defined as follows:

$$\{a_p^r, a_p^{s\dagger}\} = a_p^r a_p^{s\dagger} + a_p^{s\dagger} a_p^r$$

$r, s = \pm 1/2$
Lagrangian Density for Spin 1/2 Fermions

\[ \mathcal{L} = \bar{\Psi} \left[ i \gamma^\mu \partial_\mu - m \right] \Psi \]

Comments:

1. This Lagrangian density is used for all the quarks and leptons – only the masses will be different!

2. The Euler Lagrange equations, when applied to this Lagrangian density, give the Dirac Equation!

3. Note that \( L \) is a Lorentz scalar.
The Euler–Lagrange equations applied to $L$:

$$L = \bar{\Psi} \left[ i \gamma^\mu \partial_\mu - m \right] \Psi$$

The Euler Lagrange equations give the Dirac equation:

$$\frac{\partial L}{\partial \Psi} - \frac{\partial}{\partial x^\mu} \frac{\partial L}{\partial \left[ \partial_\mu \Psi \right]} = 0$$

$$-\bar{\Psi}m - \frac{\partial}{\partial x^\mu} \bar{\Psi} i \gamma^\mu = 0$$

$$-\Psi^\dagger \gamma^0 m - \frac{\partial}{\partial x^\mu} \Psi^\dagger \gamma^0 i \gamma^\mu = 0$$

Take the Hermitian conjugate (remember to interchange order of matrices):

$$-\gamma^0 \gamma^\dagger m + \frac{\partial}{\partial x^\mu} i \gamma^\mu \gamma^0 + \Psi = 0$$

$$\gamma^0 \gamma^\dagger = \gamma^0; \quad \gamma^\dagger k = -\gamma^k; \quad \gamma^k \gamma^0 = -\gamma^0 \gamma^k$$

Multiply on left by $\gamma^0$ and use $\gamma^0 \gamma^0 = 1$

$$-m \Psi + i \gamma^\mu \frac{\partial}{\partial x^\mu} \Psi = 0$$

$$i \gamma^\mu \partial_\mu \Psi - m \Psi = 0$$

Dirac equation

$$\left[ i \gamma^\mu \partial_\mu - m \cdot 1 \right] \Psi(r, t) = 0$$
Lagrangian Density for Spin 1/2
Quarks and Leptons

\[ L = \bar{\Psi} \left[ i \gamma^\mu \partial_\mu - m \right] \Psi \]

Now we are ready to talk about the gauge invariance that leads to the Standard Model and all its interactions. Remember a “gauge invariance” is the invariance of the above Lagrangian under transformations like \( \Psi \rightarrow e^{i\alpha} \Psi \). The physics is in the \( \alpha \) -- which can be a matrix operator and depend on \( x, y, z \) and \( t \).