DETERMINING WHETHER THE COLUMNS OF A MATRIX ARE LINEARLY DEPENDENT

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Problem: Let \( A = \begin{bmatrix} 1 & 7 & 6 \\ 2 & 6 & 4 \\ 4 & 3 & -1 \end{bmatrix} \). Determine whether the columns of \( A \) are linearly dependent or independent.

Solution:

Let \( \vec{a}_1, \vec{a}_2, \vec{a}_3 \) denote the columns of \( A \), in order. These vectors are linearly dependent if and only if there is a nonzero vector \( \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \) such that

\[
x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 = \vec{0}.
\]

They are linearly independent if and only if the condition

\[
x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 = \vec{0}
\]

implies

\[
x_1 = x_2 = x_3 = 0.
\]

Now, the equation \( x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 = \vec{0} \) is (equivalent to) the homogeneous system with \( A \) as its coefficient matrix:

\[
A\vec{x} = \vec{0}.
\]
Thus, to determine whether the columns of $A$ are linearly dependent, we will perform admissible row operations on the matrix $A$, to row-reduce it to reduced row echelon form:

\[
\begin{align*}
R_1 & \\
\frac{R_2 - 2R_1}{A} & \rightarrow \begin{bmatrix} 1 & 7 & 6 \\ 0 & -8 & -8 \\ 0 & -25 & -25 \end{bmatrix} \\
R_3 - 4R_1 & \rightarrow \begin{bmatrix} 1 & 7 & 6 \\ 0 & -8 & -8 \\ 0 & -25 & -25 \end{bmatrix} = \\
\frac{7}{8}R_2 & \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.
\end{align*}
\]

It follows that the system $A\vec{x} = \vec{0}$ is equivalent to the following simpler system:

\[
\begin{align*}
x_1 - x_3 &= 0 \\
x_2 + x_3 &= 0 \\
0 &= 0.
\end{align*}
\]

In this system, the variable $x_3$ is free, so a nontrivial solution of the system $A\vec{x} = \vec{0}$ may be obtained by substituting any nonzero number for $x_3$, and backsolving then for $x_2$ and $x_1$. For example, a nontrivial solution of the system $A\vec{x} = \vec{0}$ is given by

\[
\vec{x} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}.
\]

Hence the columns of $A$ are linearly dependent. Note that if we use the above solution of the system $A\vec{x} = \vec{0}$, we may express $\vec{a}_3$ as a linear combination of $\vec{a}_1$ and $\vec{a}_2$, as follows:

\[
\vec{a}_3 = -\vec{a}_1 + \vec{a}_2.
\]