**EIGENVALUES OF A LINEAR OPERATOR**

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**Problem:** Let \( q : \mathbb{R} \to \mathbb{R} \) be the polynomial function defined by

\[
q(x) = 1 + x,
\]

and define \( T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{P}_2(\mathbb{R}) \) by the formula

\[
T(p) = qp'.
\]

Find the eigenvalues of \( T \).

**Solution:**

First, let us verify that \( T \) is indeed a linear operator. To see this, let

\( f, g \in \mathbb{P}_2(\mathbb{R}) \), and let \( c \in \mathbb{R} \). Then

\[
T(cf + g) = (q)(cf + g)' = (q)(cf' + g') = cqf' + qg' = cT(f) + T(g),
\]

so that \( T \) is linear. Since differentiation lowers the degree of a nonzero polynomial by 1 and multiplication by \( q \) raises the degree of a nonzero polynomial by 1, it is clear that \( T \) maps \( \mathbb{P}_2(\mathbb{R}) \) into itself, so \( T \) is a linear operator. Now, let \( \mathcal{B} = (p_0, p_1, p_2) \) be the standard ordered basis for \( \mathbb{P}_2(\mathbb{R}) \):

For each \( j \), let \( p_j(x) = x^j \). We have

\[
T(p_0) = (q)p_0' = (q)(0) = 0, \quad T(p_1) = (q)p_1' = (p_0 + p_1)p_0 = p_0 + p_1,
\]

and

\[
T(p_2) = (q)p_2' = (p_0 + p_1)(2p_1) = 2p_1 + 2p_2.
\]
so that the matrix that represents $T$ relative to the basis $B$ is given by

$$[T]_B = \begin{bmatrix} [T(p_0)]_B & [T(p_1)]_B & [T(p_2)]_B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$ 

It follows that the eigenvalues of $T$ are 0, 1 and 2.