The Matrix Equation \( Ax = \vec{b} \)

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Abstract. We describe illustrate the connection between systems of equations and matrix-vector equations by discussing them and presenting some examples.

If \( A \) is a matrix whose columns are \( \vec{a}_1, ..., \vec{a}_n \), then for any vector

\[
\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},
\]

in \( \mathbb{R}^{(n \times 1)} \), the matrix product \( A\vec{x} \) is given by

\[
A\vec{x} = \sum_{j=1}^{n} x_j \vec{a}_j = x_1 \vec{a}_1 + ... + x_n \vec{a}_n.
\]

That is,

\[
\begin{bmatrix} \vec{a}_1 & ... & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{a}_1 + ... + x_n \vec{a}_n.
\]

Examples:

\[
(1) \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}
\]

Key words and phrases: linear equation, system of equations.
Given an \((m \times n)\) matrix \(A\) and an \((m \times 1)\) column vector \(\vec{b}\), the system with augmented matrix \([A|\vec{b}]\) is equivalent to the matrix-vector equation

\[
A\vec{x} = \vec{b}.
\]

This observation will give us some alternate ways to solve systems of equations and it will enable various algebraic manipulations to study systems of equations and their solutions, among other things.

**Examples:**

(1) The matrix-vector equation

\[
\begin{bmatrix}
1 & 2 \\
0 & 3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
1 \\
3
\end{bmatrix}
\]

is equivalent to the system

\[
x_1 + 2x_2 = 1
\]

\[
3x_2 = 3.
\]

(2) The system

\[
x_1 + 3x_2 - x_3 = -3
\]

\[
x_2 + 4x_3 = 17
\]

\[
2x_1 + 2x_2 + 5x_3 = 18
\]
is equivalent to the matrix-vector equation

\[
\begin{bmatrix}
1 & 3 & -1 \\
0 & 1 & 4 \\
2 & 2 & 5 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
-3 \\
17 \\
18 \\
\end{bmatrix}
\]

It can be shown that

\[
\begin{bmatrix}
-2 \\
1 \\
4 \\
\end{bmatrix}
\]

is a solution of the above matrix equation, and so it is a solution of the preceding system.

An equation of the form \( A\vec{x} = \vec{b} \) has a solution if and only if the vector \( \vec{b} \) is a linear combination of the columns of the matrix \( A \). That is, if \( A = [a_1 ... a_n] \), then the equation \( A\vec{x} = \vec{b} \) has a solution if and only if

\[
\vec{b} \in \text{Span} \{a_1, ..., a_n\}.
\]

Also, for an \((m \times n)\) matrix \( A \), the following are equivalent:

1. For each column vector \( \vec{b} \in \mathbb{R}^{(m \times 1)} \), the matrix-vector equation \( A\vec{x} = \vec{b} \) has a solution.
2. The columns of \( A \) span \( \mathbb{R}^{(m \times 1)} \).
3. Each row of \( A \) contains a pivot entry.

Examples:

1. If \( A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \), then \( \text{rref}(A) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \), so that \( A \) has no pivot position in its second row. It follows that there are some equations of the form \( A\vec{x} = \vec{b} \) that have no solution.
(2) If \( A = \begin{bmatrix}
1 & 2 & 2 \\
3 & 1 & -1 \\
4 & 3 & 1
\end{bmatrix} \), then the third row of \( A \) has no pivot entry, so for some vector \( \vec{b} \), the equation \( A\vec{x} = \vec{b} \) has no solution.

The following theorem describes some of the most useful algebraic properties enjoyed by matrix-vector products.

**Theorem:** Let \( A \in \mathbb{R}^{(m \times n)} \) and let \( \vec{u}, \vec{v} \in \mathbb{R}^{(n \times 1)} \). If \( c \in \mathbb{R} \), then \( A (c\vec{u} + \vec{v}) = cA\vec{u} + A\vec{v} \). Equivalently, \( A (\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} \) and \( A (c\vec{u}) = cA\vec{u} \).

**Examples:**

1. If \( \vec{u} = -\vec{v} \), then
   \[
   A (\vec{u}) + A (-\vec{v}) = A (\vec{u} + (-\vec{v})) = A (-\vec{v} + (-\vec{v})) = A (-2\vec{v}) = -2A\vec{v}
   \]
   and
   \[
   A\vec{u} + A\vec{v} = A (-\vec{v}) + A\vec{v} = A (-\vec{v} + \vec{v}) = A\vec{0} = \vec{0}.
   \]

2. \( A (\vec{u} + 2\vec{v}) = A\vec{u} + 2A\vec{v} \).

3. \( A (5\vec{u} + 3\vec{v}) = 5A\vec{u} + 3A\vec{v} \).

**References**


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