Math 308 - Linear Algebra II

Required Text:
Linear Algebra (2nd Ed.)
- by Kenneth Hoffman and Ray Kunze
Pub. by Prentice Hall, Inc.,
Englewood Cliffs, N.J.

Recommended Texts for Review:
1. Elementary Linear Algebra (5th Ed.)
   - by Bernard Kolman
   New York, N.Y.

2. Introduction to Linear Algebra
   - by L. W. Johnson and R. D. Riess
   Reading, Mass.
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Other Recommended Reading:
1. Schaum’s Outline for Linear Algebra
   - by S. Lipschutz
   Pub. by McGraw-Hill.

2. A First Course in Abstract Algebra (3rd Ed.)
   - by John B. Fraleigh
   Reading, Mass.
Chapter 0: ‘Naive’ Set Theory

The material discussed here is included in the first three sections of the appendix to the text. The remainder of the appendix will be presented as the need arises.

Section A.1: Sets

It is assumed that the reader has a sufficient intuitive understanding of the notion of a ‘set’, ‘class’, ‘collection’, or ‘family’, the notion of an ‘object’, and the notion of an object ‘being in’ a set, class, etc. The words ‘set’, ‘class’, ‘collection’, and ‘family’ are to be used synonymously. We assume that the reader is familiar with the natural numbers 0, 1, 2, 3, etc., and understands what it means to say that a set S has n objects in it, for some natural number n. Finally, we also assume that the notion of a ‘property’ is understood, and that it is clear whether or not a given object x ‘has’ property \( \varphi \). (In case x has property \( \varphi \), we write \( \varphi(x) \), or say that \( \varphi(x) \) is true.) It is with these understandings that the following notions and terms are defined.

Definitions:
1. If S is a set and x is an object in S, then we shall say that x \emph{belongs to} S, x \emph{is a member of} S, or that x \emph{is an element of} S, and in this case we write \( x \in S \). Otherwise we write \( x \notin S \).
2. If S is a set and for some natural number n, the objects in S are exactly the objects \( x_1, \ldots, x_n \), then we write \( S = \{ x_1, \ldots, x_n \} \), and say that the set S \emph{equals} the set \( \{ x_1, \ldots, x_n \} \). That is, we denote the (only) set whose members are \( x_1, \ldots, x_n \) by \( \{ x_1, \ldots, x_n \} \). Such a set S is said to be \emph{finite}. For \( n = 1 \), we call \( \{ x_1 \} \) a \emph{singleton} set, and for \( n = 2 \), we call \( \{ x_1, x_2 \} \) a \emph{doublet} set. If S is a set which is not finite, then we say that S is \emph{infinite}.
3. If S and T are sets such that each member of S is a member of T, then we say that the set S is \emph{contained in} the set T, that T \emph{contains} S, or that S is a \emph{subset of} T, and we write either \( S \subseteq T \) or \( T \supseteq S \) to denote this situation.
4. If S and T are sets with \( S \subseteq T \) and \( T \subseteq S \), then we say that S and T are \emph{equal} sets (because they have the same objects in them), and we denote this situation by \( S = T \). Otherwise we write \( S \neq T \).
5. If \( S \subseteq T \) but \( S \neq T \), then we say that S is \emph{properly contained in} T, or that S is a \emph{proper subset of} T, and this is denoted by \( S \subset T \) or by \( T \supset S \).
6. If \( \varphi \) is a property and if there is a set S such that S is the set of all objects x which have property \( \varphi \), then we write \( S = \{ x | \varphi(x) \} \).
7. If S and T are sets, \( \varphi \) is a property and S is the set of all objects x in T which have property \( \varphi \), then we write \( S = \{ x \in T | \varphi(x) \} \).

Examples:
1. There is no set $S$ consisting of exactly those sets which are not members of themselves, for let $\varphi$ be the property ‘$x$ is a set and $x \notin x$’. If such a set $S$ existed, then it would be given by $S = \{ x \mid \varphi(x) \}$. Suppose $S$ is a member of itself, i.e. that $S$ does not have property $\varphi$. Then $S$ is an object which does not have property $\varphi$, and so is not in $S$. This is a contradiction, because we assumed that $S$ is a member of itself. Thus $S$ cannot be a member of itself. But then $S$ has property $\varphi$, and so is a member of itself. Again this is a contradiction. Since $S$ must either be a member of itself or not, and either assumption leads to a contradiction, it must be the case that no such set exists. (This is the resolution of what is called Russell’s Paradox in Naive set theory.)

2. The following sets are understood to be given:
   (a) The empty set, $\emptyset$ has no members. It is also denoted by $\emptyset$.
   (b) The set $\mathbb{N}$ of natural numbers: $\mathbb{N} = \{ 0, 1, 2, 3, \ldots \}$.
   (c) The set $\mathbb{Z}$ of integers: $\mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \}$.
   (d) The set $\mathbb{Q}$ of rational numbers: $\mathbb{Q} = \{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \}$.
   (e) The set $\mathbb{C}$ of complex numbers: $\mathbb{C} = \{ a + bi \mid a, b \in \mathbb{R} \}$.

3. The empty set is a subset of every set, i.e. if $S$ is any set whatsoever, then $\emptyset \subseteq S$.

4. We have the following series of containments:
   $\emptyset \subset \{ 0 \} \subset \{ 0, 1 \} \subset \ldots \subset \{ 0, \ldots, n \} \subset \ldots \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

5. If we denote the set of all nonnegative integers by $\mathbb{Z}^0$, then we have $\mathbb{Z}^0 \subseteq \mathbb{N}$ and $\mathbb{N} \subseteq \mathbb{Z}^0$, so that $\mathbb{N} = \mathbb{Z}^0$, but if we denote the set of all positive integers by $\mathbb{Z}^+$, then we have $\mathbb{Z}^+ \subseteq \mathbb{N}$ but not $\mathbb{N} \subseteq \mathbb{Z}^+$, so $\mathbb{Z}^+ \neq \mathbb{N}$. We will prefer the notation $\mathbb{N}$ over the notation $\mathbb{Z}^0$, and we will prefer to write $\mathbb{N} \setminus \{ 0 \}$ rather than $\mathbb{Z}^+$. 

Definitions:
1. If $S$ and $T$ are sets, then the union of $S$ and $T$ is the set $S \cup T$ defined by the property ‘$x \in S$ or $x \in T$’; that is,
   $$ S \cup T = \{ x \mid x \in S \text{ or } x \in T \}. $$

2. If $S$ and $T$ are sets, then the intersection of $S$ and $T$ is the set $S \cap T$ defined by the property ‘$x \in S$ and $x \in T$’; that is,
   $$ S \cap T = \{ x \mid x \in S \text{ and } x \in T \} = \{ x \in S \mid x \in T \}. $$

3. If $S$ and $T$ are sets, then the complement of $T$ in $S$ is the set $S \setminus T$ defined by the property ‘$x \in S$ and $x \notin T$’; that is,
   $$ S \setminus T = \{ x \mid x \in S \text{ and } x \notin T \} = \{ x \in S \mid x \notin T \}. $$

4. If $S_1, \ldots , S_n$ are sets and $T$ is the union of the first $n$-1 of these, then we define the union of all $n$ of them to be $T \cup S_n$: 

\[
\bigcup_{k=1}^{n} S_k = S_1 \cup \ldots \cup S_n = T \cup S_n = ( \bigcup_{k=1}^{n-1} S_k ) \cup S_n.
\]

5. If \( S_1, \ldots, S_n \) are sets and \( T \) is the intersection of the first \( n-1 \) of these, then we define the \textit{intersection} of all \( n \) of them to be \( T \cap S_n \):

\[
\bigcap_{k=1}^{n} S_k = S_1 \cap \ldots \cap S_n = T \cap S_n = ( \bigcap_{k=1}^{n-1} S_k ) \cap S_n.
\]

6. If \( S \) is a family of sets, then the \textit{union} of the sets in \( S \) is the set \( \bigcup S \) defined by the property ‘\( x \in S \), for some \( S \in S \)’, i.e. we have:

\[
\bigcup S = \{ x \mid x \in S, \text{ for some } S \in S \}.
\]

We also denote \( \bigcup S \) by \( \bigcup_{S \in S} S \), or, if \( I \) is a set such that \( S = \{ S_i \mid i \in I \} \), then we denote \( \bigcup S \) by \( \bigcup_{i \in I} S_i \).

7. If \( S \) is a family of sets, then the \textit{intersection} of the sets in \( S \) is the set \( \bigcap S \) defined by the property ‘\( x \in S \), for every \( S \in S \)’, i.e. we have:

\[
\bigcap S = \{ x \mid x \in S, \text{ for every } S \in S \}.
\]

We also denote \( \bigcap S \) by \( \bigcap_{S \in S} S \), or, if \( I \) is a set such that \( S = \{ S_i \mid i \in I \} \), then we denote \( \bigcap S \) by \( \bigcap_{i \in I} S_i \).

Examples:
1. Let \( S_1 = \{ 1, 2, 3 \}, S_2 = \{ 3, 4, 5 \} \), \( S = \{ S_1, S_2 \} \). Then we have the following:

\[
\bigcup S = S_1 \cup S_2 = \bigcup_{i \in \{ 1, 2 \}} S_i = \{ 1, 2, 3, 4, 5 \},
\]

and similarly,

\[
\bigcap S = S_1 \cap S_2 = \bigcap_{i \in \{ 1, 2 \}} S_i = \{ 3 \}.
\]

2. Let \( S \) be the set of all intervals of real numbers of length less than 1 which include the numbers 0 and 0.5. Then we have

\[
\bigcup S = (-0.5, 1) = \{ x \in \mathbb{R} \mid -0.5 < x < 1 \}
\]

and

\[
\bigcap S = [0, 0.5] = \{ x \in \mathbb{R} \mid 0 \leq x \leq 0.5 \}.
\]