Partial Differential Equations (PDE) (CLW: 7.1, 7.3, 7.4)

PDE’s can be linear or nonlinear
Order : Determined by the order of the highest derivative.
Linear, 2nd order PDE’s are classified as the elliptic, hyperbolic
Parabolic type.

Example:

\[
A_1 \frac{\partial^2 u}{\partial x^2} + A_2 \frac{\partial^2 u}{\partial y^2} + A_3 \frac{\partial^2 u}{\partial z^2} + B_1 \frac{\partial u}{\partial x} \\
+ B_2 \frac{\partial u}{\partial y} + B_3 \frac{\partial u}{\partial z} + C u + D = 0
\]
Coefficients $A_1, A_2, A_3$ May be +1, -1 or zero.

$u$ is the dependent variable and $x, y, z$ are the independent variables. Note that we do not have any cross-derivative terms.

Classification:

Elliptic: $A_1, A_2, A_3$ are non-zero and have the same sign, then PDE is of the elliptic type.

Hyperbolic: $A_1, A_2, A_3$ are non-zero and of mixed sign, the PDE is hyperbolic

Parabolic: If one of $A_1, A_2, A_3$, say $A_2$ is zero and the rest are of the same sign, and if $B_2$ is non-zero, the PDE is parabolic.

Since $A_i, B_i, C$ and $D$ may be functions of $x, y$ and $z$, the classification may depend on position in space.

In many CFD problems, one of the independent variables will be time and the rest will be space coordinates such as $x, y, z$ or transformed variables such as $\xi, \eta, \zeta$. 
Two-Dimensional Examples

Elliptic
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0
\]

Hyperbolic
\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}
\]

Parabolic
\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}
\]

Numerical solution of PDE requires a finite number of points to discretize the equations.
Examples:

See Figure in the next slide.
Solution requires initial and boundary conditions depending on the Problem.
Indices $i, j, n$ can be used to label the nodes in $x, y, t$ directions (see fig.)
If the origin has $i=0, j=0$ and $n = 0$, then the node $i, j, n$ has coordinates $\Delta x, \Delta y, n\Delta t$,
where $\Delta x, \Delta y, \Delta t$ are the uniform intervals between nodes along $x, y, t$ coordinate directions.
Let $u(x,y,t) \equiv u_{i,j,n}$ be the exact solution of the PDE and $v_{i,j,n}$ be the approximations to be determined at each grid point.
The derivatives of the original PDE are approximated using the symbol $V_{i,j,n}$ and the discretization intervals $\Delta x, \Delta y, \Delta t$.

The procedure leads to a set of algebraic equations of which are then solved.

Fine grids can be used to obtain solutions $V_{i,j,n}$ that are close to $U_{i,j,n}$.

Examples of PDE’s common in engineering

1. Unsteady heat conduction equation

1D form:

$$\frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) = c_p \rho \frac{\partial T}{\partial t}$$

If $k$ is a constant, the equation becomes

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

Where $\alpha = \frac{k}{c_p \rho}$ is the thermal diffusivity.
In CFD, normalization of variables are often used to improve the solution (for proper scaling of the variables).

Let \( \xi = \frac{x}{L}, \tau = \frac{\alpha t}{L^2} \)

for heat conduction in a rod of length \( L \).

The PDE then becomes

\[
\frac{\partial^2 T}{\partial \xi^2} = \frac{\partial T}{\partial \tau}
\]

It is also possible to non-dimensionalize the dependent variable \( T \).

Taylor’s Expansion.

Let \( \frac{dy}{dx} = f(x, y) \). Taylor series is as follows.

\[
y(x_0 + h) = y(x_0) + h f(x_0, y(x_0)) + \frac{h^2}{2!} f'(x_0, y(x_0)) + \frac{h^3}{3!} f''(x_0, y(x_0)) + \ldots
\]

Where

\[
f'(x, y(x)) = \frac{df}{dx}(x, y(x))
\]

\[
f''(x, y(x)) = \frac{d^2f}{dx^2}(x, y(x))
\]
The higher order derivatives in Eq. (2) can be determined by differentiating Eq. (1) by chain rule. i.e.,

\[
\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}
\]

Example 1: \( f(x,y) \) is a function of \( x \) alone.

\[
\frac{dy}{dx} = x^2
\]

\[
f^{(i)}(x,y) = 2x \quad f^{(i)}(x_0,y_0) = 2x_0
\]

\[
f^{(ii)}(x,y) = 2 \quad f^{(ii)}(x_0,y_0) = 2
\]

\[
f^{(iii)}(x,y) = 0 \quad f^{(iii)}(x_0,y_0) = 0
\]

\[
...........................
\]

\[
f^{(n)}(x,y) = 0 \quad f^{(n)}(x_0,y_0) = 0
\]
The function at the neighboring point \( (x = x_0 + h) \) becomes
\[
y(x_0 + h) = y(x_0) + hx_0^2 + h^2x_0 + \frac{h^3}{3}
\]

Example 2: \( f(x,y) \) is a function of \( y \) alone.
\[
\frac{dy}{dx} = f(x,y) = 2y
\]

\[
f'(x,y) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = 0 + 2 \times 2y = 4y
\]

Similarly
\[
f''(x,y) = 8y
\]
\[
f'''(x,y) = 16y
\]

\[
f^n(x,y) = 2^{(n+1)}y
\]
Taylor series can also be used for non-linear higher order equations. Example:

\[
\frac{d^2 y}{dx^2} - \frac{dy}{dx} + xy^2 = 0
\]

\[
\frac{dy}{dx} = f(x, y) = \frac{d^2 y}{dx^2} + xy^2
\]

\[
y'' = y' - xy^2
\]

\[
y''' = y'' - 2xyy' - y^2
\]
\[ y^{iv} = y''' - (2yy' + 2xyy'' + 2y^2') - 2yy' \]
\[ y^{iv} = y''' - 4yy' - 2y^2'' - 2xyy'' \]

Consider the initial conditions

At \( x = 0, y(0) = 1, y'(0) = -1 \)

\[ y'''(0) = y'(0) - 0 \times y^2(0) = -1 \]
\[ y''''(0) = y''(0) - y^2(0) \]
\[ y''''(0) = -1 - 1 = -2 \]
\[ y^{iv}(0) = y'''(0) - 4 \times 1(-1) \]
\[ = -2 + 4 = 2 \]
Since Taylor series gives
\[ y_{k+1} = y_k + y_k' \frac{h}{1!} + y_k'' \frac{h^2}{2!} + y_k''' \frac{h^3}{3!} + \ldots \]

For \( k = 0 \), \( y_1 \) becomes
\[ y_1' = 1 - h - \frac{h^2}{2} - \frac{h^3}{3} + \frac{h^4}{12} + \ldots \]

Letting \( h = 0.1 \), we get
\begin{align*}
y_1' &= 1 - 0.1 - \frac{0.01}{2} - \frac{0.001}{3} + \frac{0.0001}{12} \\
y_1 &= 0.894675
\end{align*}

Calculation of \( y_2 \)

Need \( y_1'' \)

Calculate \( y_1'' \) by differentiating the expression for \( y_1' \) w.r.t. \( h \).
\[ y_1'' = -1 - h - h^2 + \frac{h^3}{3} + \ldots \]
\[ y_1'' = -1 - 0.1 - 0.01 + \frac{0.001}{3} = -1.1097 \]

\( y_1''', y_1'''' \) etc. can now be calculated as before and \( y_2 \) can be obtained.