Quantum Two
Motion in 3D as a Direct Product
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each of which is describes a quantum particle moving along one of the three orthogonal Cartesian axes.
For a particle moving along each axis, there is / are different state space basis vectors & operators of interest:

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$$|x, y, z\rangle = |x\rangle \otimes |y\rangle \otimes |z\rangle = |r\rangle$$

each labeled by 3 Cartesian coordinates of the position vectors of $R^3$. 
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Thus, each state $|\psi\rangle$ is represented in this basis by a wave function

$$\psi(\vec{r}) = \psi(x, y, z)$$

of these three independent (quantum) mechanical degrees of freedom.
Note that by forming the space as a direct product, the individual components of the position operator $X, Y, Z$ automatically commute with one another, since they come from different factor spaces.
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Indeed, the canonical commutation relations

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\[ X\left| x, y, z \right> = X\left| x \right> \mathbf{1}_y \left| y \right> \mathbf{1}_z \left| z \right> = x\left| x, y, z \right> \]
It is left to the viewer to verify that all other properties of the space of a single particle moving in 3 dimensions follow entirely from the properties of the direct product of 3 one-dimensional factor spaces.
State Space of a Spin $\frac{1}{2}$ Particle
as a Direct Product Space
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It is a well-established experimental fact that the quantum states of most fundamental particles are **not** completely specified by properties related simply to **their motion through space**.

In general, each quantum particle possesses an **internal structure** characterized by a vector observable \( \hat{S} \), the components \( S_x, S_y, S_z \) of which **transform under rotations like the components of angular momentum**.
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Such a particle is said to possess **spin degrees of freedom**.
For the constituents of atoms (electrons, neutrons, protons) the internal state of each particle can be represented as a superposition of two orthonormal eigenvectors

\[ \{|s\rangle\} = \{|1/2\rangle, \|-1/2\rangle\} \]

of the operator \( S_z \) with eigenvalues \( s = +1/2 \) and \( s = -1/2 \) (in units of \( \hbar \)).
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These two orthogonal internal states of the particle span a 2-dimensional state space, \( S_{\text{spin}} \) referred to as the particle’s **spin space**.
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The spin-degrees of freedom are assumed to be independent of the spatial degrees of freedom of the center of mass, as with, e.g., a classical spinning top.
The main point here, of course, is that the **total state space** of a spin-1/2 particle can then be represented as a direct product

\[ S_{\text{spin-1/2}} = S_{\text{spatial}} \otimes S_{\text{spin}} \]

of the quantum space describing **the particle's translational motion** (spanned by the position eigenstates \( |\vec{r}\rangle \), and the 2-dimensional spin space spanned by the spin eigenstates \( |s\rangle \) of \( S_z \).
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\langle \vec{r}', s'|\vec{r}, s\rangle = \delta(\vec{r} - \vec{r}')\delta_{s,s'} \quad \text{and} \quad \sum_{s=\pm 1/2} \int d^3r \ |\vec{r}, s\rangle \langle \vec{r}, s| = 1
\]
An arbitrary state of a spin-1/2 particle can then be expanded in this basis in the usual way

\[ |\psi\rangle = \sum_{s=\pm 1/2} \int d^3r \ \psi_s(\vec{r}) |\vec{r}, s\rangle \]
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\[ |\psi_+ (\vec{r})|^2 \text{ gives the probability density to find the particle spin-up at } \vec{r} \]

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Note also that, through the rules of the inner product, all spin operators $\vec{S}, S_z, S^2, \ldots$ automatically commute with all spatial operators. Thus, the concept of a direct product space arises in many different situations in quantum mechanics. When such a structure is properly identified it can help elucidate the structure of the underlying state space.
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a description of the state space, possible basis vectors, and operators of interest, for a system of N particles of various types (spinless or not) moving in different numbers of possible spatial dimensions.