The Fourier Series is but one example of the expansion of a periodic function as a series of orthonormal functions, $\phi[n]$, which may be complex. In order to represent a localized, non-repetitive function, the Fourier Transform must be used. This procedure can be developed from the Fourier Series by letting the period go to infinity.

$$F(x) = \sum_n c_n \phi_n$$

$$F(x) = \sum_n c_n \frac{1}{\sqrt{\lambda}} e^{jk_n x}$$

$$c_n = \frac{1}{\sqrt{\lambda}} \int_{-\lambda/2}^{\lambda/2} e^{-(j.k_n x)} F(x) \, dx$$

Where: $k_n = n \frac{2\pi}{\lambda}$

and $\delta k = \frac{2\pi}{\lambda}$

which is the difference between allowed values of $k$

As $\lambda$ goes to $\infty$ in order to make $F(x)$ have one repetition, $\delta k$, a finite difference, goes to $dk$, an infinitesimal.

$$dk = \frac{2\pi}{\lambda}$$

or

$$\frac{1}{\lambda} = \frac{dk}{2\pi}$$

After substituting for $c$ in the series representation of $F(x)$, and using normalized plane waves, we get

$$F(x) = \sum_n \frac{1}{\lambda} \left[ \int_{-5\lambda}^{5\lambda} e^{-(j.k_n x)} F(x) \, dx \right] e^{j.k_n x}$$

Substituting $dk/2\pi$ for $1/\lambda$, and letting the summation become an integral ($\lambda$ goes to $\infty$, $kn$ becomes $k$) gives

$$F(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \left[ \int_{-\infty}^{\infty} e^{-(j.k x)} F(x) \, dx \right] e^{j.k x} \, dk$$

This integral form for the representation of $F(x)$ as a linear combination (infinite sum of infinitessimals) can be represented as the Fourier Transform integrals.

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{j.k x} \, dk$$

Sum an infinite number of plane waves with amplitudes $\phi(k)$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{-j.k x} \, dx$$

Find the coefficient function, $\phi(k)$, by projecting the function $F(x)$ onto the infinite orthonormal set $\phi(k)$. Remember that $\phi(k)$ has the complex conjugate plane wave $\exp(-jkx)$. 
EXAMPLE: UNIT SQUARE PULSE FROM -b TO b

\[ \phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{-jkx} \, dx \]

\[ \phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-b}^{b} e^{-jkx} \, dx \]

\[ \phi(k) = \frac{2b}{\sqrt{2\pi}} \frac{e^{jkb} - e^{-(jk)b}}{2jkb} \]

\[ \phi(k) := \frac{2}{\pi b} \sin(kb) \frac{1}{kb} \]

\[ \phi(k) = \frac{2}{\pi} b \cdot \text{sinc}(kb) \]

Check to see if the function is reproduced. We will approximate the integral by a summation to speed the calculation and to improve the convergence.

\[ k := -K..K \]

\[ F(x) := \frac{1}{\pi} + \frac{1}{\sqrt{2\pi}} \sum_k \phi(k) e^{jkx} \]

\[ x := -2,-1.8..2 \]

(Change the value of K to explore)