Lecture 14: Gravitational potential energy and space travel

- Universal gravitational potential energy
- Space travel problems
- Escape speed
- Orbital energy
- Multiple objects
Gravitational potential energy

\[ F_{\text{grav}} = \frac{G m M}{r^2}, \text{ attractive} \]

Conservative force

\[ U_B - U_A = -W_{A\rightarrow B} = - \int_{\hat{r}_A}^{\hat{r}_B} \vec{F} \cdot d\vec{r} \]
\[ U_B - U_A = -W_{A\rightarrow B} = -\int_{\vec{r}_A}^{\vec{r}_B} \vec{F} \cdot d\vec{r} \]

\[ \int_{\vec{r}_A}^{\vec{r}_B} \vec{F} \cdot d\vec{r} = \int_{r_A}^{r_B} F_r dr = \int_{r_A}^{r_B} -\frac{GmM}{r^2} dr \]

\[ = -\left[ -\frac{GmM}{r} \right]_{r_A}^{r_B} = \frac{GmM}{r_B} - \frac{GmM}{r_A} \]

\[ U_B - U_A = -GmM \left( \frac{1}{r_B} - \frac{1}{r_A} \right) \]

Choose reference point: \( r_0 = \infty \).
Assign \( U(r_0 = \infty) = 0 \)

\[ U_{\text{Grav}} = -\frac{GmM}{r} \]

negative!
Potential energy diagram

\[ U(r_0 = \infty) = 0 \]

\[ U_G = -\frac{GMm}{r} \]

\[ E < 0 \] Bound orbits

Hyperbolic trajectory
Potential energy diagram

\[ U_G = -\frac{GMm}{r} \]

parabolic orbit \( U = 0 \) @ \( r = \infty \)
Escape condition

\[ E = \frac{1}{2}mv^2 - \frac{GmM}{r} \]

Critical escape condition: *barely* making it to \( r = \text{infinity} \), having slowed to speed of zero
**Escape speed**

**Minimum speed** an object must have at distance $R$ from central mass $M$ if it is to go infinitely far away

$$E_i = E_f$$

$$\frac{1}{2}mv_{esc}^2 - \frac{GmM}{R} = \frac{1}{2}mv_{\infty}^2 - \frac{GmM}{r = \infty}$$

$$\frac{1}{2}mv_{esc}^2 - \frac{GmM}{R} = 0 - 0$$

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$
Example: Escape speed from Earth

\[ M_{Earth} = 5.97 \times 10^{24} \text{kg} \]
\[ R_{Earth} = 6.38 \times 10^6 \text{m} \]

\[ G = 6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2 \]

\[ v_{esc} = \sqrt{\frac{2GM}{R}} = 11,200 \text{ m/s} \]
Example: Escape speed from orbit

Speed necessary to escape gravitational field of the Sun when object is launched from Earth:

\[ M_{Sun} = 1.99 \times 10^{30} \text{ kg} \]
\[ R_{SE} = 1.50 \times 10^{11} \text{ m} \]
\[ G = 6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2 \]

\[ v_{esc} = \sqrt{\frac{2GM_{Sun}}{R_{SE}}} = 42,000 \text{ m/s} \]
Orbital Energy

\[ E = K + U = \frac{1}{2}mv^2 - \frac{GmM}{R} \]

Speed of satellite in circular orbit: \( v^2 = \frac{GM}{R} \)

\[ E = \frac{1}{2}m \left( \frac{GM}{R} \right) - \frac{GmM}{R} \]

\[ E = -\frac{GmM}{2R} \]
Satellite Motion

\[ \Sigma F_x = F_{grav,x} = m_{sat} a_x = m_{sat} (+a_C) \]

\[ \frac{G m_{sat} M}{R^2} = m_{sat} \frac{v^2}{R} \]

\[ \frac{GM}{R} = v^2 \]
Multiple objects

\[ U_g = U_g_1 + U_g_2 = -\frac{GM_1m}{r_1} + \left(-\frac{GM_2m}{r_2}\right) \]
Example with multiple objects

A planet has mass $4M$ and a radius $2r$. Its moon has mass $M$ and radius $r$. The centers of the planet and moon are a distance $9r$ apart.

A shuttle of mass $m$ is a distance $4r$ away from the center of the planet and moving with speed $V$.

What is the total mechanical energy of the shuttle?
If the shuttle was initially at rest at X, how much work did the engines do?