Lecture 23: Simple Harmonic Motion

- Motion of a mass at the end of a spring
- Differential equation for simple harmonic oscillation
- Amplitude, period, frequency and angular frequency
- Energetics
Mass at the end of a spring

Mass $m$ connected to a spring with spring constant $k$ on a frictionless surface

$$F_x = -kx$$

Linear restoring spring force
Spring force: \( F_S x = -kx \)

- \( x = l - l_{eq} \)
  - stretch or compression
  - \( k \) force constant

\( F_x \) is negative if \( x \) is positive
  - (stretched spring)

\( F_x \) is positive if \( x \) is negative
  - (compressed spring)
Newton’s 2\textsuperscript{nd} Law: \( \sum F_x = m a_x \)

\[-kx = m \frac{d^2 x}{d t^2} \]

\[-\frac{k}{m} x = \frac{d^2 x}{d t^2} \]

\[
\frac{d^2 x}{d t^2} = -\omega^2 x \quad * 
\]

\[
\omega = \sqrt{\frac{k}{m}} 
\]

Differential equation of a Simple Harmonic Oscillator

*We can always write it like this because m and k are positive.*
Solution

\[ \frac{d^2x}{dt^2} = -\omega^2 x \]

Equation for SHO

General solution: \[ x = A \cos(\omega t + \varphi) \]

\[ \frac{dx}{dt} = -A \omega \sin(\omega t + \varphi) \]
\[ \frac{d^2x}{dt^2} = -A \omega^2 \cos(\omega t + \varphi) = -\omega^2 x \]

\( A \) and \( \varphi \): two “constants of integration” from solution of a second-order differential equation. Determined by the initial conditions.
Amplitude

\[ x = A \cos(\omega t + \varphi) \]

Range of cosine function: -1…+1
\[ \Rightarrow -A \leq x(t) \leq +A \]

\[ A = \text{Amplitude of the oscillation} \]
If $\varphi=0$:

$$x = A \cos(\omega t)$$

$$x(t = 0) = x_0 = A$$

To describe motion with different starting points:
Add phase constant to shift the cosine function
\[ x = A \cos(\omega t + \varphi) \]

\[ x_0 = -A : \] shift by \( \pi \)

\[ x_0 = 0 : \] shift by \( \frac{\pi}{2} \)
Initial conditions

\[ x_0 = x(t = 0) \]
\[ v_{x0} = v_x(t = 0) \]

\[ x_0 = A \cos(0 + \phi) = A \cos(\phi) \]
\[ v_{x0} = -A \omega \sin(0 + \phi) = -A \omega \sin(\phi) \]

→ two equations for \( A \) and \( \phi \)
Position and velocity

\[ x = A \cos(\omega t + \varphi) \]

\[ v_x = \frac{dx}{dt} = -A\omega \sin(\omega t + \varphi) \]

At time \( t_m \): \( x = x_{max} = A \quad \cos(\omega t_m + \varphi) = 1 \)

\( (\omega t_m + \varphi) = 0 \) or \( \pi \)

\( \sin(\omega t_m + \varphi) = 0 \quad \Rightarrow \quad v_x(t_m) = 0 \)

Mass stops and reverses direction when it reaches maximum displacement (turning point)
Simulation

http://www.walter-fendt.de/ph14e/springpendulum.htm
Period and angular frequency

\[ x = A \cos(\omega t + \varphi) \]

Time \( T \) for one complete cycle: period

(\( \omega t + \varphi \)) changes by \( 2\pi \) in time \( T \)

\[ \omega T = 2\pi \implies \omega = \frac{2\pi}{T} = 2\pi f \]
Effect of mass and amplitude on period

\[ \omega T = 2\pi \quad \Rightarrow \quad T = \frac{2\pi}{\omega} \]

\[ \omega = \sqrt{\frac{k}{m}} \quad \Rightarrow \quad T = \frac{2\pi}{\sqrt{\frac{k}{m}}} \]

\[ T = 2\pi \sqrt{\frac{m}{k}} \]

Amplitude \( A \) does not appear – no effect on period

Demo: Vertical springs showing effect of \( m \) and \( A \)
Potential energy of spring force: \( U = \frac{1}{2} k x^2 \)

At \( x = \pm A \):

\[
U = \frac{1}{2} k A^2
\]

\( K = 0 \)

\( E = \frac{1}{2} k A^2 \)

At \( x = 0 \):

\( U = 0 \)

\( K = K_{\text{max}} = E \)
Kinetic and potential energy in SHO

\[ K = \frac{1}{2} m v^2 = \frac{1}{2} m [A \omega \sin(\omega t + \phi)]^2 \]

\[ K_{\text{max}} = \frac{1}{2} m v_{\text{max}}^2 = \frac{1}{2} m (\omega A)^2 \]

\[ U = \frac{1}{2} k x^2 = \frac{1}{2} k [A \cos(\omega t + \phi)]^2 \]

\[ U_{\text{max}} = \frac{1}{2} k x_{\text{max}}^2 = \frac{1}{2} k A^2 \]

\[ E = K_{\text{max}} \sin^2(\omega t + \phi) + U_{\text{max}} \cos^2(\omega t + \phi) \]

http://www.walter-fendt.de/ph14e/springpendulum.htm
Example

A block of mass $M$ is attached to a spring and executes simple harmonic motion of amplitude $A$. At what displacement(s) $x$ from equilibrium does its kinetic energy equal twice its potential energy?