Lecture 24: General Oscillations

- Simple pendulum
- Physical pendulum
- Diatomic molecule
- Damped oscillations
- Driven oscillations
Equation for SHO

General solution:

\[ x = A \cos(\omega t + \varphi) \]

\[ T = \frac{2\pi}{\omega} \]
Simple Pendulum

Point mass $m$ at the end of a massless string of length $L$

$\theta = \text{displacement coordinate (with sign) from vertical equilibrium position}$
Simple Pendulum

\[ \Sigma \tau_z = I \alpha_z \]
\[ -mg \ L \sin \theta = mL^2 \frac{d^2 \theta}{dt^2} \]

Very complicated differential equation!

But for small oscillations:
\[ \sin \theta \approx \theta \]

And
\[ -\frac{g}{L} \theta = \frac{d^2 \theta}{dt^2} \]

Differential equation of SHO
Simple pendulum oscillations

\[-\frac{g}{L} \theta = \frac{d^2 \theta}{dt^2}\]  
Differential equation of simple harmonic oscillator

\[\theta(t) = \theta_{max} \cos(\omega t + \varphi)\]

With \[\omega = \sqrt{\frac{g}{L}}\] and \[T = 2\pi \sqrt{\frac{L}{g}}\]

Demo: Simple pendulum with different masses, lengths and amplitudes
Demo: Simple pendulum with different masses, lengths and amplitudes

- Period independent of mass
- Period independent of amplitude
Extended object of mass $m$ that swings back and forth about an axis $P$ that does not go through its center of mass $\text{CM}$.

\[ \Sigma \tau_z = I \alpha_z \]

\[ -mgD \sin \theta = I \frac{d^2 \theta}{dt^2} \]

For small oscillations: $\sin \theta \approx \theta$

\[ -\frac{mgD}{I} \theta = \frac{d^2 \theta}{dt^2} \]

SHO
Motion of the Physical Pendulum

\[-\frac{mgD}{I} \theta = \frac{d^2 \theta}{dt^2}\]

SHO

\[\theta = \Theta_{\text{max}} \cos(\omega t + \varphi)\]

\[\omega = \sqrt{\frac{mgD}{I}}\]

\[T = 2\pi \sqrt{\frac{I}{mgD}}\]

$I$ is moment of inertia about axis $P$

$D$ is distance between $P$ and $CM$

Parallel axis theorem:

\[I_P = I_{CM} + mD^2\]

Demo: Meter stick pivoted at different positions
\[ T = 2\pi \sqrt{\frac{I}{mgD}} \]
Example

A uniform disk of mass $M$ and radius $R$ is pivoted at a point at the rim. Find the period for small oscillations.

$T = 2\pi \sqrt{\frac{I}{mgD}}$

$I_P = I_{CM} + mD^2$
Diatomc molecule: two atoms separated a distance $r$
Diatomic molecule near equilibrium

Near equilibrium (potential minimum):
$U$ approximately quadratic
$F$ approximately linear

Expansion about $r_{eq}$:

$r = r_{eq} + x$

for small displacement $x$ from equilibrium:

$$U(x) \approx U_{min} + \frac{1}{2}kx^2$$
$$F_x(x) \approx -kx$$

$\Rightarrow$ harmonic oscillation about equilibrium position
For homework problem #4:

**Binomial Theorem**

\[(1 + u)^n = 1 + n u + \frac{n(n-1)}{2!} u^2 + \ldots\]

If \(u \ll 1\):

\[(1 + u)^n \approx 1 + nu\]

\[T = 2 \pi \sqrt{\frac{m}{k}}\]
Damped Oscillations

Damping force proportional to velocity \( F_{Dx} = -bv_x \)

Newton’s 2nd Law: \( \sum F_x = m\ddot{x} \)

\[
-kx - bv_x = m \frac{d^2x}{dt^2}
\]

\[
\frac{d^2x}{dt^2} = - \left[ \frac{k}{m} x - \frac{b}{m} \frac{dx}{dt} \right]_{\omega_0^2}
\]

\[
x(t) = A_0 e^{-\frac{bt}{2m}} \cos(\omega t + \varphi) \quad \text{with} \quad \omega^2 = \omega_0^2 - \frac{b^2}{4m^2}
\]
Characteristics of damped oscillation

\[ x(t) = A_0 e^{-\frac{bt}{2m}} \cos(\omega t + \varphi) \]

Amplitude decays exponentially. With

\[ \omega^2 = \omega_0^2 - \frac{b^2}{4m^2} \]

If \( \omega < \omega_0 \)

Critical damping:

\[ \text{if } \omega_0 = \frac{b}{2m} \]

\[ \omega = 0 \]

No oscillation
Driven oscillations

Oscillating system with natural frequency $\omega_0$

Externally applied periodic driving force $F_d = F_0 \cos(\omega t)$

\[ m \frac{d^2x}{dt^2} = -kx - b \frac{dx}{dt} + F_0 \cos \omega t \]

\[ x = A_0 \sin(\omega t + \varphi) \]

\[ A_0 = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \frac{b^2\omega^2}{m^2}}} \]

http://www.walter-fendt.de/ph14e/resonance.htm
Resonance

\[ x = A_0 \sin(\omega t + \varphi) \]

\[ A_0 = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \frac{b^2 \omega^2}{m^2}}} \]

If driving frequency close to natural frequency \( \omega \approx \omega_0 \) catastrophic growth of amplitude = Resonance

http://www.walter-fendt.de/ph14e/resonance.htm
Millennium Bridge, London

Video: Spontaneous resonance
Tacoma Narrows Bridge Collapse

Cause still disputed, not technically resonance – but the video is worth watching and fits very well here