Physics 481: Condensed Matter Physics - Homework 8

due date: March 18, 2011

Problem 1: Damped oscillations (10 points)

In a linear chain of lattice spacing $a$, particles of mass $m$ are connected by nearest-neighbor springs of spring constant $K$. In addition to the elastic forces, each particle is subjected to a damping force $F_D = -\Gamma \dot{u}_n$, where $u_n$ is the displacement of the $n$th particle from the equilibrium position. How does the damping change the frequencies $\omega(k)$, and what is the relaxation time of the modes? Assume $(\Gamma/m)^2 \ll K/m$ and discuss the limiting cases $k \approx 0$ and $k \approx \pi/a$.

Problem 2: Creation and destruction operators (15 points)

Consider a quantum-mechanical harmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2M} + \frac{1}{2} M \omega^2 x^2$$

where $M$ is the mass, $\omega$ is the frequency and $p$ and $x$ and the momentum and position operators fulfilling the commutator $[p, x] = \hbar/i$. The destruction and creation operators are defined by

$$a = \sqrt{\frac{M \omega}{2\hbar}} x + \frac{i}{\sqrt{2M \omega \hbar}} p$$

$$a^\dagger = \sqrt{\frac{M \omega}{2\hbar}} x - \frac{i}{\sqrt{2M \omega \hbar}} p$$

a) Calculate the commutator $[a, a^\dagger]$.

b) Show that the Hamiltonian can be written as $H = \hbar \omega (a^\dagger a + 1/2)$.

c) $|n\rangle$ denotes the normalized eigenstate with energy $E_n = \hbar \omega (n + 1/2)$. Show that $a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$ and $a |n\rangle = \sqrt{n} |n-1\rangle$.

Problem 3: Low-temperature specific heat in $d$ dimensions and for nonlinear dispersion laws (Ashcroft-Mermin problem 23.2, 15 points)

Consider small lattice vibrations in a $d$-dimensional crystal in harmonic approximation.

a) For the Debye model, i.e. a linear dispersion $\omega = c|k|$ of all phonon modes, calculate the phonon density of states and show that it varies as $\omega^{d-1}$. What is the Debye frequency?

b) Determine the phonon contribution to low-temperature specific heat.

c) Investigate what would happen for a nonlinear phonon dispersion $\omega \sim |k|^\nu$ (anomalous sound). Show that the low-temperature specific heat would vanish as $T^{d/\nu}$ in $d$ dimensions.