Problem 1: Spin-3/2 paramagnet (80 points)

A paramagnetic material contains \( N \) localized (and thus distinguishable), non-interacting \( S = \frac{3}{2} \) quantum spins in a magnetic field \( \vec{B} = Be_\mathbf{z} \) in z-direction. The Hamiltonian is given by

\[
H = -\mu B \sum_{i=1}^{N} S_i^z,
\]

where \( S_i^z \) has the eigenvalues \(-3/2, -1/2, 1/2, 3/2\). Here, \( \mu \) is the magnetic moment associated with the spins.

a) Qualitatively discuss the ground state. What is the ground state energy \( E_0 \)? What is the ground state magnetization \( M_0 = \mu \sum_{i=1}^{N} S_i^z \)? (10 points)

b) Use the canonical ensemble to calculate the partition function and the Helmholtz free energy. (20 points)

c) Compute the internal energy as a function of temperature and discuss its behavior in the limits \( k_B T \ll \mu B \) and \( k_B T \gg \mu B \). Compare with the ground state energy you found in a). (20 points)

d) Calculate the magnetization \( M = \mu \sum_{i=1}^{N} \langle S_i^z \rangle \) as function of temperature and field. Discuss its behavior in the limits \( k_B T \ll \mu B \) and \( k_B T \gg \mu B \). Compare with the ground state magnetization you found in a). (20 points)

e) Find the magnetic susceptibility \( \chi = (\partial M/\partial B)_T \) in the high temperature limit \( k_B T \gg \mu B \) and determine the Curie constant. [Hint: You can simplify the calculation by taking the high-temperature limit before calculating the field-derivative.] (10 points)

Problem 2: Debye phonons in one dimension (80 points)

Consider a one-dimensional solid of \( N_A \) atoms and length \( L \). This solid has longitudinal phonons but no transversal ones. Within the Debye model, the phonons have frequencies \( \omega_k = c|k| \) (where \( c \) is the sound velocity and \( k \) is the wave number). Phonon modes only exist for frequencies below the Debye frequency \( \Omega_D \).

a) Calculate the density of states \( g(\epsilon) \). (20 points)

b) What is the total number of phonon modes in a chain of \( N_A \) atoms? Use this to determine the Debye frequency \( \Omega_D \) as a function of \( N_A \) and \( L \). (20 points)

c) Derive an expression for the internal energy as a functions of temperature. (20 points)

d) Discuss the internal energy in the limits of high \( (k_B T \gg \hbar \Omega_D) \) and low \( (k_B T \ll \hbar \Omega_D) \) temperatures. Compare with the equipartition theorem. (20 points)
Problem 3: Mean-field theory of an Ising model with long-range interactions (90 points + 25 BONUS)

Consider a one-dimensional Ising model of $N$ spins, $S_i = \pm 1$, given by a Hamiltonian

$$H = -J \sum_{i \neq j} \frac{1}{(i-j)^2} S_i S_j - \mu B \sum_i S_i$$

with a positive interaction constant $J$ (i.e., the interaction is not just between nearest neighbors but between all pairs; its magnitude decays quadratically with distance). Assume $N \gg 1$ and periodic boundary conditions.

a) Qualitatively discuss the ground state in the absence of a field ($B = 0$)? What is its energy? (10 points)

b) Derive the mean-field Hamiltonian for this model. (20 points)

c) Solve the mean-field Hamiltonian and derive the mean-field equation for the magnetization. (20 points)

d) Find the critical temperature for the onset of magnetic order in the absence of a field. (20 points)

e) Determine the magnetic susceptibility above the critical temperature in the limit of small fields. Is the Weiss temperature positive or negative. (20 points)

f) [25 BONUS points] Consider the case $J < 0$. Qualitatively describe the ground state and calculate its energy. Derive the mean-field theory for the type of order you have identified. Find the critical temperature.

---

\[
\int_0^\infty dx \frac{x}{e^x - 1} = \frac{\pi^2}{6}.
\]

\[
tanh(x) = x - \frac{1}{3}x^3 + \ldots, \quad \text{Artanh}(x) = x + \frac{1}{3}x^3 + \ldots
\]

\[
\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^\infty \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}
\]