Problem 1: Shannon entropy of independent random variables (8 points)

Consider the joint probability distribution $P_{X_1 \ldots X_N}$ of $N$ random variables $X_1, \ldots, X_N$. Show that if the $X_i$ are statistically independent then the Shannon entropy $S$ of their joint distribution is the sum of the Shannon entropies of the reduced distributions $P_{X_i}$ of the individual variables $X_i$.

Problem 2: Shannon entropy of $N$ spin-2 atoms (5 points)

Consider a lattice with $N \gg 1$ spin-2 atoms. Each atom can be in one of the five spin states $S_z = -2, -1, 0, +1, +2$ with equal probability. Calculate the Shannon entropy of this system.

Problem 3: Maxima of entropy (12 points)

Consider the entropy of a discrete probability distribution given in terms of the probabilities $p_i$ ($i = 1 \ldots N$). Determine which $p_i$ lead to the maximum entropy under the following constraints (Hint: Use Lagrange multipliers to enforce the constraints.):

a) Normalization $\sum_i p_i = 1$

b) Normalization $\sum_i p_i = 1$ and fixed average $\langle a \rangle = \sum_i p_i a_i$ of a quantity $A$ with values $a_i$.

c) Normalization, fixed average $\langle a \rangle$ and fixed variance $\sigma^2_A$.

Problem 4: Atoms on a lattice (15 points)

Consider a lattice having $N$ regular lattice sites as well as $N$ interstitial lattice sites. The lattice is occupied by $N$ identical atoms. An atom on a regular site has energy 0 while an atom on an interstitial site has energy $\epsilon$. Use the microcanonical ensemble to calculate the internal energy and the specific heat as functions of temperature.