Physics 6311: Statistical Mechanics - Homework 5

due date: Friday, Mar 4, 2016

Problem 1: Relativistic ideal gas in microcanonical ensemble (12 points)

Consider a gas of \( N \gg 1 \) relativistic, non-interacting, distinguishable particles in a cubic box of linear size \( L \). The energy-momentum relation of a single particle is \( \epsilon = c|\vec{p}| \).

a) Calculate the entropy as function of the energy.
b) Calculate the temperature and the caloric equation of state (energy-temperature relation).
c) Calculate the thermodynamic equation of state (relation between \( p, V, T \)).
d) Show that the ratio of the specific heats \( C_p/C_V = 4/3 \) (and not 5/3 as in the nonrelativistic case).

Problem 2: Spin-\( \frac{1}{2} \) in a magnetic field (12 points)

Consider \( N \) (distinguishable) \( S = \frac{1}{2} \) quantum spins in a magnetic field \( B = B\mathbf{e}_z \) in \( z \)-direction. The Hamiltonian is given by

\[
\hat{H} = -B g \mu_B \sum_{i=1}^{N} S_i^{(z)}, \quad S_i^{(z)} = \pm \frac{1}{2}.
\]

Here \( g \) is the gyromagnetic factor and \( \mu_B \) is the Bohr magneton. The spins are coupled to a heat bath at temperature \( T \).

a) Use the canonical ensemble to calculate the partition function, Helmholtz free energy, the entropy, the internal energy and the specific heat as functions of temperature.
b) Calculate the magnetization \( M = g \mu_B \sum_{i=1}^{N} \langle S_i^{(z)} \rangle \) and the magnetic susceptibility \( \chi = (\partial M/\partial B)_T \) as functions of \( T \) and \( B \). Determine the zero-field susceptibility \( \lim_{B \to 0} \chi(B, T) \).
c) For fixed \( B > 0 \), at what temperature is the maximum of the specific heat (the so-called Schottky peak)?

Problem 3: Specific heat of an anharmonic oscillator (16 points)

An anharmonic classical oscillator has the Hamiltonian

\[
H = \frac{p^2}{2m} + V_0 \cosh(x/x_0)
\]

where the mass \( m \) as well as the potential parameters \( V_0 \) and \( x_0 \) are constants.
a) Calculate the specific heat as a power series in the temperature $T$ to linear order in $T$. (To decide how to set up the expansion, think about where the particle will be at low and high temperatures, respectively. What does this mean for the potential?)

b) Also calculate the contribution of order $T^2$. 