Problem 1: Relativistic ideal gas

a) eigenstates are plane waves $\psi_k = V^{-1/2} e^{ik \cdot x}$ with wavevectors $k_i = (2\pi/L)n_i$ with $n_i$ integer

$$\epsilon = c|p| = ch \sqrt{k_x^2 + k_y^2 + k_z^2}$$

Total energy

$$E = ch \sum_{i=1}^{N} \sqrt{(k_{i,x}^2 + k_{i,y}^2 + k_{i,z}^2)}$$

To count microstates we need the volume of the object defined by this equation in 3$N$-dimensional $k$-space. The length scale of the object is $R_k = E/(ch)$. Thus, the volume is $V_k = B(N) [E/(ch)]^{3N}$ where the constant prefactor only depends on $N$.

The number of states with energies below $E$, is therefore

$$\Sigma(E, N, V) = V_k/(2\pi/L)^{3N} = B(N) \left( \frac{LE}{ch} \right)^{3N}$$

$$\Omega(E, N, V) \sim \partial \Sigma(E, N, V) / \partial E \sim \frac{3N}{E} \Sigma(E, N, V)$$

In calculating $S$ we use the fact that it has to be extensive to pull the appropriate factors of $N$ into the logarithm:

$$S(E, N, V) = Nk_B \ln \left[ \frac{V}{N} \left( \frac{E}{Nch} \right)^3 \right] + f(N)$$

b) Solve for the energy:

$$E = Nch(N/V)^{1/3} e^{\frac{S-f(N)}{3Nk_B}}$$

temperature $T = (\partial E/\partial S)_{N,V} = E/(3Nk_B)$, therefore $E = 3Nk_BT$.

c) pressure $p = -\left( \partial E/\partial V \right)_{N,S} = E/(3V) = Nk_BT/V$, therefore $pV = Nk_BT$.

d) $C_p - C_v = TV\alpha^2/\kappa_T$

The r.h.s. only depends on the thermodynamic eq. of state, thus it is the same as for the non-relativistic ideal gas: $C_p - C_v = Nk_B$

Therefore: $C_p = 4Nk_B$ and $C_p/C_v = 4/3$. 
Spin 1/2 in a magnetic field

\[ a) \quad Q = \sum \frac{e^{-\beta H}}{s_z} = \sum \frac{e^{-\beta g_0 m_z}}{s_z} \]

\[ Q = 2 \cosh \left( \frac{1}{2} \beta g_0 m_0 \right) \]

\[ A = -k_B T \ln Q = -k_B T \ln \left[ 2 \cosh \left( \frac{1}{2} \beta g_0 m_0 \right) \right] \]

\[ \bar{E} = -\frac{2}{\partial \beta} \ln \frac{Q}{Z} = -\frac{2}{\partial \beta} \ln \left[ 2 \cosh \left( \frac{1}{2} \beta g_0 m_0 \right) \right] \]

\[ \bar{E} = -\frac{1}{2} g_0 m_0 + \tan \left( \frac{1}{2} \beta g_0 m_0 \right) \]

\[ S = \frac{1}{T} (\bar{E} - A) = -k_B \frac{1}{2} g_0 m_0 + \tan \left( \frac{1}{2} \beta g_0 m_0 \right) \]

\[ + k_B \ln \left[ 2 \cosh \left( \frac{1}{2} \beta g_0 m_0 \right) \right] \]

\[ C = \frac{\partial S}{\partial T} = T \frac{2}{\bar{E}} \left( \frac{1}{T} (\bar{E} - A) \right) = -\frac{1}{T} (\bar{E} - A) + \frac{3}{T} \frac{\partial T}{\partial \bar{E}} \bar{E} = -\frac{2}{\partial T} A \]

\[ C = \frac{\partial \bar{E}}{\partial T} = \left( \frac{1}{2} g_0 m_0 \right) \frac{1}{\cosh^2 \left( \frac{1}{2} \beta g_0 m_0 \right)} \]

\[ C = k_B \left( \frac{g_0 m_0}{2 k_B T} \right)^2 \frac{1}{\cos \left( \frac{1}{2} \beta g_0 m_0 \right)} \]
3) \[ m = g \mu_3 \langle s_z \rangle = \frac{1}{2} g \mu_3 \tanh \left( \frac{1}{2} \beta B g \mu_3 \right) \]

\[ \chi = \left( \frac{\partial m}{\partial B} \right)_T = \left( \frac{1}{2} g \mu_B \right)^2 \beta \frac{1}{\cosh^2 \left( \frac{1}{2} \beta B g \mu_3 \right)} \]

\[ \lim_{B \to 0} \chi = \left( \frac{1}{2} g \mu_B \right)^2 \frac{1}{k_B T} \]

Carrie law

- \[ C = k_B x^2 / \cosh^2 \left( x \right) \]

\[ \lim_{T \to 0} : x \to \infty \quad C \text{ vanishes exponentially} \]

\[ \lim_{T \to \infty} : x \to 0 \quad C \text{ vanishes geometrically} \]

\[ \text{Maximum at} \quad D = \frac{\partial C}{\partial x} = \frac{2x}{\cosh^2 \left( x \right)} - \frac{2x^2 \sinh \left( x \right)}{\cosh^2 \left( x \right)} \]

\[ 1 = x \tanh \left( x \right) \quad \text{at} \quad x = 1.20 \]

\[ k_0 T_{\text{max}} = \frac{1}{2} \beta B g \mu_3 / 1.20 \]
Problem 3: Specific heat of an anharmonic oscillator

At low temperatures, the particle is close to the minimum of the potential at \( x = 0 \). We can therefore expand the potential in a power series about \( x = 0 \). The kinetic energy is quadratic in the momentum, it thus contributes \( k_B/2 \) to the specific heat, and we can focus on the positional part of \( Q \).

\[
Q_{pot} = \int_{-\infty}^{\infty} dx \exp[-\beta V_0 \cosh(x/x_0)] = \int_{-\infty}^{\infty} dx \exp[-\beta V_0 (1+x^2/(2x_0^2)+x^4/(4! x_0^4)+x^6/(6! x_0^6)+O(x^8))] 
\]

The quadratic term restricts the \( x \) values to \( |x| \lesssim x_{\text{max}} = \sqrt{x_0^2/\beta} \). Therefore, the higher order terms in the expansion are of order \( T \) or smaller for small \( T \), and we can expand the exponentials of these terms.

\[
Q_{pot} \sim \int_{-\infty}^{\infty} dx \exp[-\beta V_0 x^2/(2x_0^2)] \{ 1 - \beta V_0 x^4/(4! x_0^4) - \beta V_0 x^6/(6! x_0^6) + (\beta V_0 x^4/(4! x_0^4))^2/2 + O(\beta x^8) \} = \\
= \sqrt{2\pi x_0^2/\beta V_0} \left[ 1 - 3/(4! \beta V_0) - 15/(6! \beta^2 V_0^2) + 105/[2 \cdot 4! \beta^2 V_0^2] \right] \\
\ln Q_{pot} = -1/2 \ln \beta + \ln \left[ 1 - 1/(8\beta V_0) + 9/(128\beta^2 V_0^2) \right] 
\]

For small \( T \), the log can be expanded \( \ln(1 + x) = 1 + x - x^2/2 + .... \)

\[
\ln Q_{pot} = -1/2 \ln \beta - 1/(8\beta V_0) + 1/(16\beta^2 V_0^2) 
\]

\[
\langle E_{pot} \rangle = -\frac{\partial \ln Q_{pot}}{\partial \beta} = 1/(2\beta) - 1/(8\beta^2 V_0) + 1/(8\beta^3 V_0^2) 
\]

\[
C = \frac{\partial \langle E \rangle}{\partial T} = \frac{k_B}{2} + \frac{k_B}{2} - \frac{k_B T}{4V_0} + \frac{3k_B^2 T^2}{8V_0^2} 
\]