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RICCATI INEQUALITY, DISCONJUGACY, AND RECIPROCITY PRINCIPLE FOR LINEAR HAMILTONIAN DYNAMIC SYSTEMS

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ABSTRACT. We investigate linear Hamiltonian dynamic systems (H). In particular, we characterize the disconjugacy of (H) in terms of the solvability of a Riccati inequality. This generalizes the known result of linear Hamiltonian differential systems and yields a new result for the discrete case. Further, we investigate a connection between eventual disconjugacy of (H) and its reciprocal system.

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1. INTRODUCTION

Disconjugacy of linear Hamiltonian differential systems

$$(\mathrm{H}_C) \qquad x' = A(t)x + B(t)u, \quad u' = C(t)x - A^T(t)u,$$

where $t \in [a,b] \subseteq \mathbb{R}$, $B(t), C(t)$ are symmetric $n \times n$-matrices and $B(t) \geq 0$, has been studied in several monographs and many papers, see e.g., [13, 17, 24, 26, 27, 28]. The subscript in (H_C) and elsewhere refers to the "continuous-time" case. The well-known result [26, Theorem 5.1], now called a Reid roundabout theorem, states that the disconjugacy of (H_C) on the interval $[a,b]$ is equivalent to the positivity of the quadratic functional

$$\mathcal{F}_C(x,u) := \int_a^b \left\{ x^T(t)C(t)x(t) + u^T(t)B(t)u(t) \right\} dt,$$

and to the existence of a symmetric solution of the Riccati matrix differential equation

$$(\mathrm{R}_C) \qquad R_C[W](t) := W' - C(t) + A^T(t)W + WA(t) + WB(t)W = 0.$$

By using the Sturm comparison theorem, Reid proved in his book [26, Theorem 5.3] or [27, pg. 140], see also [13, pg. 62], that actually the existence of a symmetric solution to the Riccati *inequality* $R_C[W](t) \leq 0$ is a necessary and sufficient condition for (H_C) to be disconjugate on $[a,b]$.

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In [8], Bohner presented a discrete version of the Reid roundabout theorem concerning the linear Hamiltonian difference system

$$(\mathrm{H}_D) \qquad \Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k,$$

where $k \in [0, N] \subseteq \mathbb{Z}$, B_k, C_k are symmetric $n \times n$-matrices, and $I - A_k$ is invertible. The subscript in (H_D) and elsewhere refers to the "discrete-time" case. In particular, disconjugacy of (H_D) was defined in an appropriate way and characterized in [8] in terms of the positivity of the discrete quadratic functional

$$\mathcal{F}_D(x, u) := \sum_{k=0}^{N} \left\{ x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k \right\},$$

and in [9] in terms of the solvability of the associated Riccati matrix difference equation

$$(\mathrm{R}_D) \qquad R_D[W]_k := \Delta W_k - C_k + A_k^T W_k + (W_{k+1} - C_k) \tilde{A}_k (A_k + B_k W_k) = 0,$$

where $\tilde{A}_k := (I - A_k)^{-1}$. According to our knowledge, the corresponding discrete-time result on the Riccati *inequality* is not known, except of the special cases of scalar second order equations [5, Eq. (2.7')] and [4, Theorem 5.1], second order matrix equations [6, Theorem 3.1], and symmetric three term recurrences [2, Remark 8]. The reason is perhaps the fact that the discrete Riccati operator $R_D[W]_k$ is *not symmetric*, as opposed to the continuous-time Riccati operator $R_C[W]$. Thus, the inequality $R_D[W]_k \leq 0$ does not make sense.

In this paper, we approach this problem via the investigation of Hamiltonian *dynamic* systems, i.e., linear Hamiltonian systems

$$(\mathrm{H}) \qquad x^\Delta = A(t) x^\sigma + B(t) u, \quad u^\Delta = C(t) x^\sigma - A^T(t) u$$

on an arbitrary time scale $\mathbb{T}$, which is a nonempty closed subset of $\mathbb{R}$, see [12, 18]. Here $B(t), C(t)$ are symmetric $n \times n$-matrices and $I - \mu(t) A(t)$ is invertible, where $\mu(t)$ is the graininess of the time scale $\mathbb{T}$, see the next section. Disconjugacy of (H) was defined by the first author in [19]. Its characterization in terms of the positivity of the corresponding quadratic functional

$$\mathcal{F}(x, u) := \int_a^b \left\{ (x^\sigma)^T C x^\sigma + u^T B u \right\} (t) \, \Delta t$$

was obtained in [20], and in terms of the solvability of the Riccati matrix dynamic equation

$$(\mathrm{R}) \quad R[W](t) := W^\Delta - C(t) + A^T(t) W + [W^\sigma - \mu(t) C(t)] \tilde{A}(t) [A(t) + B(t) W] = 0,$$

in [21], where $\tilde{A}(t) := [I - \mu(t) A(t)]^{-1}$. As in the discrete case, the Riccati operator $R[W]$ is not symmetric. Instead, we show that the matrix $(X^\sigma)^T R[W] X$ is symmetric, where X is an invertible $n \times n$-matrix-valued function satisfying $X^\Delta = A(t) X^\sigma +$

$B(t)U$. This implies that the time scales version of the Riccati inequality takes the form

$$\tilde{A}^T(t)\, R[W](t)\, [I + \mu(t)B(t)W]^{-1} \le 0. \tag{1.1}$$

Now, since $\mu(t) \equiv 0$ in the continuous case $\mathbb{T} = \mathbb{R}$, we have $\tilde{A}(t) \equiv I$ and inequality (1.1) becomes $R_C[W](t) \le 0$ in this case. On the other hand, if $\mathbb{T} = \mathbb{Z}$, then the desired discrete Riccati inequality is

$$\tilde{A}_k^T R_D[W]_k (I + B_k W_k)^{-1} \le 0. \tag{1.2}$$

Thus, this is the situation when the time scales theory yields a new result even for the discrete-time theory.

The so-called *reciprocity principle* plays an important role in the theory of linear Hamiltonian differential (as well as difference) systems. It says that if $B(t) \ge 0$, $C(t) \le 0$, and both systems (H_C) and

$$y' = -A^T(t)y - C(t)z, \quad z' = -B(t)y + A(t)z \tag{$\tilde{\mathrm{H}}_C$}$$

satisfy a certain assumption of eventual controllability, then (H_C) is eventually disconjugate iff ($\tilde{\mathrm{H}}_C$) is eventually disconjugate, see [1, 25]. The reciprocity principle is of particular importance when the systems (H_C) and ($\tilde{\mathrm{H}}_C$) correspond to the self-adjoint differential equations

$$(-1)^n \left(r(t)y^{(n)}\right)^{(n)} = p(t)y \quad \text{and} \quad (-1)^n \left(\frac{1}{p(t)}\, y^{(n)}\right)^{(n)} = \frac{1}{r(t)}\, y,$$

respectively, where $p(t), r(t) > 0$. Moreover, this statement has many applications in the spectral theory of singular differential operators. The reciprocity principle for linear Hamiltonian difference systems was proved in [11, Theorem 3], see also [15, Theorem 5.1]. Note that the reciprocal system to (H_D) takes the form

$$\Delta y_k = -A_k^T y_k - C_k z_{k+1}, \quad \Delta z_k = -B_k y_k + A_k z_{k+1}.$$

In this paper, we unify and extend the reciprocity principle to the linear Hamiltonian dynamic system (H) and its reciprocal counterpart

$$y^\Delta = -A^T(t)y - C(t)z^\sigma, \quad z^\Delta = -B(t)y + A(t)z^\sigma. \tag{$\tilde{\mathrm{H}}$}$$

In fact, the reciprocal system ($\tilde{\mathrm{H}}$) is here defined as the system which results from (H) upon the transformation $y = -u$, $z = x$. This transformation just reverses the order of equations in (H) (and also moves the jump operator to the second component z). Consequently, the reciprocity principle says, roughly speaking, that (H) is oscillatory with respect to the first component x iff it is oscillatory with respect to the second component u.

The paper is organized as follows. In the next section we recall some basic properties of the linear Hamiltonian dynamic systems. Then, in Section 3, we state and prove our first main result, which relates the disconjugacy of (H) with the solvability of the Riccati dynamic inequality (1.1). In Section 4, we specialize to the discrete case $\mathbb{T} = \mathbb{Z}$ and extend the discrete Riccati inequality (1.2) to problems with variable endpoints. The last section is devoted to the reciprocity principle for linear Hamiltonian dynamic systems.

2. PRELIMINARIES

Let $\mathbb{T}$ be a time scale. We assume that the reader is familiar with basic calculus on time scales [12, 18]. As usual, the *forward jump operator* is denoted by σ and then the *graininess* of $\mathbb{T}$ is $\mu(t) := \sigma(t) - t$. A composition of a function f on $\mathbb{T}$ and the forward jump operator σ is denoted by f^σ. Recall that the delta-derivative f^Δ of f is defined in such a way that $f^\Delta(t) = f'(t)$ (the usual derivative) for $\mathbb{T} = \mathbb{R}$, and $f^\Delta(t) = \Delta f_t = f_{t+1} - f_t$ (the usual forward difference operator) for $\mathbb{T} = \mathbb{Z}$. If $[a, b]$ is an interval in $\mathbb{T}$, then $[a, b]^\kappa$ denotes the interval $[a, b]$ without a possible isolated maximum.

We will use the notation $\operatorname{Ker} M$, $\operatorname{Im} M$, M^T, M^{T-1}, $M^\dagger$, $M \geq 0$, and $M > 0$ to denote the kernel, image, transpose, inverse of the transpose, Moore-Penrose generalized inverse, nonnegative definiteness, and positive definiteness of a given matrix M, respectively. The latter two properties are considered for a symmetric matrix M, i.e., $M^T = M$. We will write $M \leq 0$ for $-M$ being nonnegative definite. By I and 0 we denote the identity matrix and the zero matrix (or vector), respectively, of appropriate dimensions. All quantities in this paper are assumed to be real valued.

Remark 2.1. The following properties of delta-differentiable functions hold.

(i) Difference quotient $f^\sigma = f + \mu f^\Delta$.

(ii) Product rule $(fg)^\Delta = f^\Delta g^\sigma + f g^\Delta = f^\Delta g + f^\sigma g^\Delta$.

(iii) Inverse matrix delta-derivative. If X is delta-differentiable matrix such that both X and X^σ are invertible, then $(X^{-1})^\Delta = -X^{-1} X^\Delta (X^\sigma)^{-1} = -(X^\sigma)^{-1} X^\Delta X^{-1}$.

Let $\mathbb{T} = [a, b]$ be a time scale interval. With $n \in \mathbb{N}$, let be given rd-continuous $n \times n$-matrices A, B, C such that $B(t), C(t)$ are symmetric and $I - \mu(t)A(t)$ is invertible. We denote $\tilde{A}(t) := [I - \mu(t)A(t)]^{-1}$, which is then also rd-continuous. By a solutions of (H) or (R) we mean an rd-continuously delta-differentiable function satisfying the relevant system or equation on $[a, b]^\kappa$. Vector solutions of (H), typically (x, u), will be denoted by small letters and the $n \times n$-matrix-valued solutions of (H), typically (X, U), by capital ones.

The transition matrix from (X, U) to (X^σ, U^σ) in system (H) is a symplectic (hence invertible) matrix, see [19, Remark 4(ii)] or [7] for the discrete case. Thus, the following identities hold throughout $[a,b]^\kappa$

$$X^\sigma = \tilde{A}X + \mu\tilde{A}BU, \quad U^\sigma = \mu C\tilde{A}X + (\mu^2 C\tilde{A}B + \tilde{A}^{T-1})U, \tag{2.1}$$

$$X = (\mu^2 B\tilde{A}^T C + \tilde{A}^{-1})X^\sigma - \mu B\tilde{A}^T U^\sigma, \quad U = -\mu\tilde{A}^T C X^\sigma + \tilde{A}^T U^\sigma. \tag{2.2}$$

The invertibility of the transition matrix also implies that the initial value problems associated with (H) possess unique solutions.

Following [19], a solution (X,U) of (H) is a *conjoined basis* if $X^T(t)U(t)$ is symmetric and $\operatorname{rank}(X^T(t)\ U^T(t)) = n$ at some (and hence at any) $t \in [a,b]$. The solution (X,U) of (H) given by the initial conditions $X(a) = 0$, $U(a) = I$ is called a *principal solution.* A conjoined basis (X,U) has *no focal points* in the interval $(a,b]$ if $X(t)$ is invertible at all dense points $t \in (a,b]$, and for all $t \in [a,b]$ the following two conditions hold

$$\operatorname{Ker} X^\sigma(t) \subseteq \operatorname{Ker} X(t), \quad D(t) := X(t)[X^\sigma(t)]^\dagger \tilde{A}(t)B(t) \geq 0.$$

Recall that t is a *dense point* if it is either left-dense or right-dense. System (H) is said to be *disconjugate on* $[a,b]$ if the principal solution of (H) has no focal points in $(a,b]$.

System (H) is said to be *strongly normal* on the interval $[a,s] \subseteq [a,b]$ if s is a dense point and the only solution (x,u) of (H) for which $x(t) \equiv 0$ on $[a,s]$ is the zero solution $(x(t),u(t)) \equiv 0$ on $[a,s]$. Equivalently, system (H) is strongly normal on $[a,s]$ whenever s is a dense point and the only solution of the system

$$-u^\Delta = A^T(t)u, \quad B(t)u = 0, \quad t \in [a,s]^\kappa, \tag{2.3}$$

is the zero solution $u(t) \equiv 0$ on $[a,s]$. The hypothesis of strong normality will be denoted by the following:

(D) $\left\{\begin{array}{l}\text{System (H) is strongly normal on any interval of the form} \\ [a,s] \subseteq [a,b] \text{ for all dense points } s \in (a,b].\end{array}\right.$

Note that the disconjugacy of (H) on $[a,b]$ implies condition (D), which follows from the next lemma.

Lemma 2.2. *If the principal solution* (X,U) *of* (H) *has* $X(t)$ *invertible for all dense points* $t \in (a,b]$*, then* (D) *holds.*

Proof. Let (X,U) be the principal solution of (H), i.e., $X(a) = 0$ and $U(a) = I$. Let $s \in (a,b]$ be a fixed dense point and suppose that $u(t)$ is a nontrivial solution of (2.3) on $[a,s]^\kappa$. Then $(0,u(t))$ solves (H) on $[a,s]^\kappa$ and necessarily $u(a) \neq 0$. For if $u(a) = 0$, then $(0,u(t))$ satisfies the initial condition $(0,u(a)) = (0,0)$, which would imply $u(t) \equiv 0$ on $[a,s]$ by the uniqueness theorem, which is a contradiction. Thus, in this case of $u(a) \neq 0$, we set $(x(t),v(t)) := (X(t),U(t))u(a)$ on $[a,s]$. Then $(x(a),v(a)) = (X(a),U(a))u(a) = (0,u(a))$, so that both $(0,u(t))$ and $(x(t),v(t))$

are solutions of (H) on $[a,s]^\kappa$ satisfying the same initial condition. Consequently, $(x(t), v(t)) \equiv (0, u(t))$ on $[a,s]$. In particular, $X(t)u(a) = x(t) \equiv 0$ on $[a,s]$, which for $t = s$ yields $X(s)u(a) = 0$. This however contradicts the invertibility of $X(s)$. Therefore, $u(t) \equiv 0$ on $[a,s]$ and we showed that (D) holds. □

A pair (x,u) is called *admissible* if x is piecewise rd-continuously delta-differentiable on $[a,b]^\kappa$, u is piecewise rd-continuous on $[a,b]^\kappa$, and $x^\Delta = Ax^\sigma + Bu$ holds on $[a,b]^\kappa$ (at points $t \in [a,b]$, where x^Δ is not continuous, this is to be read as the corresponding right/left-sided limit). The quadratic functional $\mathcal{F}$ is called *positive definite*, we write $\mathcal{F} > 0$, if $\mathcal{F}(x,u) > 0$ for all admissible pairs (x,u) with $x(a) = 0 = x(b)$ and $x(t) \not\equiv 0$.

Proposition 2.3 ([19, Theorem 1])**.** *Suppose that there exists a conjoined basis of* (H) *with no focal points in* $(a,b]$. *Then* $\mathcal{F} > 0$.

The following proposition is a (slightly improved) version of the Reid roundabout theorem presented in [21, Theorem 1].

Proposition 2.4 (Reid roundabout theorem)**.** *The following are equivalent.*

(i) *System* (H) *is disconjugate on* $[a,b]$.
(ii) *The quadratic functional* $\mathcal{F}$ *is positive definite and* (D) *holds.*
(iii) *There exists a conjoined basis* (X,U) *of* (H) *with no focal points in* $(a,b]$, *such that* $X(t)$ *is invertible for all* $t \in [a,b]$, *and* (D) *holds.*
(iv) *There exists a symmetric solution* $W(t)$ *of the Riccati matrix dynamic equation* (R) *with* $I + \mu(t)B(t)W(t)$ *invertible and* $[I + \mu(t)B(t)W(t)]^{-1}B(t) \geq 0$ *on* $[a,b]^\kappa$, *and* (D) *holds.*

We will need a Sturmian-type comparison theorem with another linear Hamiltonian dynamic system

$$(\underline{\mathrm{H}}) \qquad x^\Delta = \underline{A}(t)x^\sigma + \underline{B}(t)u, \quad u^\Delta = \underline{C}(t)x^\sigma - \underline{A}(t)^T u,$$

where the coefficient matrices $\underline{A}, \underline{B}, \underline{C}$ satisfy the same general assumptions as A, B, C. Let us denote by ($\underline{\mathrm{D}}$) the corresponding dense normality condition for the system ($\underline{\mathrm{H}}$). Also, we set

$$\mathcal{H} := \begin{pmatrix} -C - A^T B^\dagger A & A^T B^\dagger \\ B^\dagger A & -B^\dagger \end{pmatrix}, \quad \underline{\mathcal{H}} := \begin{pmatrix} -\underline{C} - \underline{A}^T \underline{B}^\dagger \underline{A} & \underline{A}^T \underline{B}^\dagger \\ \underline{B}^\dagger \underline{A} & -\underline{B}^\dagger \end{pmatrix}.$$

Proposition 2.5 (Sturm comparison theorem)**.** [20, Theorem 3.8] *Let* $\underline{\mathcal{H}}(t) \leq \mathcal{H}(t)$ *and* $\operatorname{Im}(\underline{A}(t) - A(t) \quad \underline{B}(t)) \subseteq \operatorname{Im} B(t)$ *hold on* $[a,b]^\kappa$. *Suppose that the system* ($\underline{\mathrm{H}}$) *satisfies the strong normality condition* ($\underline{\mathrm{D}}$). *If there exists a conjoined basis of* (H) *with no focal points in* $(a,b]$, *then* ($\underline{\mathrm{H}}$) *is disconjugate on* $[a,b]$.

An application of the above comparison theorem to systems where $\underline{A}(t) = A(t)$ and $\underline{B}(t) = B(t)$ yields the following statement.

Corollary 2.6. *Let* $\underline{A}(t) = A(t)$, $\underline{B}(t) = B(t)$, *and* $\underline{C}(t) \geq C(t)$ *on* $[a,b]^\kappa$. *If* (H) *is disconjugate on* $[a,b]$, *then* ($\underline{\mathrm{H}}$) *is disconjugate on* $[a,b]$ *as well.*

In order to prove the results on the reciprocity principle we need to introduce the following concepts. Let $\mathbb{T}$ be unbounded above, i.e., $\sup \mathbb{T} = \infty$. In the definitions containing the word "eventually" we consider the relevant systems on the interval of the form $[a, \infty)$.

System (H) is said to be *eventually disconjugate* (or *nonoscillatory*) if there exists $t_0 \in \mathbb{T}$ such that this system is disconjugate on $[t_0, t_1]$ for every $t_1 > t_0$. In the opposite case (H) is said to be *oscillatory*. Similarly we define the eventual disconjugacy of ($\tilde{\mathrm{H}}$). More precisely, the disconjugacy of ($\tilde{\mathrm{H}}$) on $[a,b]$ means that the solution (Y,Z) of ($\tilde{\mathrm{H}}$) given by $Y(a) = 0$, $Z(a) = I$ has $Y(t)$ is invertible at all dense points $t \in (a,b]$ and for all $t \in [a,b]^\kappa$ one has

$$\operatorname{Ker} Y^\sigma(t) \subseteq \operatorname{Ker} Y(t), \quad Y^\sigma(t) Y^\dagger(t) \tilde{A}^T(t) C(t) \leq 0.$$

System (H) is said to be *eventually strongly normal* if there exists $t_0 \in \mathbb{T}$ such that this system is strongly normal on $[t_0, s]$ for every dense $s > t_0$. System (H) is said to be *eventually identically normal*, if it is eventually strongly normal and, when where is no dense point in (t_0, ∞), the following condition holds: there exists $l \in \mathbb{N}$ such that for any $t_1 \geq t_0$, if $x^{\sigma^k}(t_1) = 0$ for all $k = 0, \dots, l$, then $(x(t), u(t)) \equiv 0$ on (t_1, ∞). Here $\sigma^k = \underbrace{\sigma \circ \cdots \circ \sigma}_{k-\text{times}}$ and $\sigma^0(t) = t$. Similarly we define these concepts for ($\tilde{\mathrm{H}}$). In particular, (eventual) strong normality of ($\tilde{\mathrm{H}}$) is deduced from the system

$$z^\Delta = A(t) z^\sigma, \quad C(t) z^\sigma = 0, \quad t \in [a,s]^\kappa$$

having only the zero solution $z(t) \equiv 0$ on $[a,s]$, where $s \in \mathbb{T}$, $s > a$, is a dense point.

A conjoined basis $(\bar{X}, \bar{U})$ of (H) is said to be a *principal solution* of (H) at ∞ if $\bar{X}(t)$ is nonsingular, $[\bar{X}^\sigma(t)]^{-1} \tilde{A}(t) B(t) [\bar{X}(t)]^{T-1} \geq 0$, both for large t, and $\lim_{t\to\infty} X^{-1}(t) \bar{X}(t) = 0$ for any conjoined basis (X,U) for which the (constant) Wronskian matrix $X^T \bar{U} - U^T \bar{X}$ is nonsingular. Nonoscillatory and eventually identically normal system (H) possesses the principal solution, by [14, Theorem 3.1]. The solution $W^-(t)$ of the Riccati equation (R) generated by the principal solution $(\bar{X}, \bar{U})$ of (H) at ∞, i.e., $W^- = \bar{U}\bar{X}^{-1}$, is said to be *eventually minimal at* ∞ (another terminology is *distinguished*). Note that this definition of $W^-(t)$ implies that, eventually, $I + \mu(t) B(t) W^-(t)$ is invertible and $[I + \mu(t) B(t) W^-(t)]^{-1} B(t) \geq 0$. The solution $W^-(t)$ is an eventually minimal solution of (R) in the sense that if $W(t)$ is any solution of (R) on some interval $[t_0, \infty)$ such that $I + \mu(t) B(t) W(t)$ is invertible and satisfying $[I+\mu(t)B(t)W(t)]^{-1}B(t) \geq 0$ therein, then $W(t) \geq W^{-}(t)$ on $[t_1,\infty)$ for some t_1, see [14, Theorem 3.2].

Identities (2.1) and (2.2), when rewritten for the reciprocal system ($\tilde{\mathrm{H}}$), read as follows

$$Y^\sigma = (\mu^2 C\tilde{A}B + \tilde{A}^{T-1})Y - \mu C\tilde{A}Z, \quad Z^\sigma = -\mu\tilde{A}BY + \tilde{A}Z,$$
$$Y = \tilde{A}^T Y^\sigma + \mu\tilde{A}^T C Z^\sigma, \quad Z = \mu B\tilde{A}^T Y^\sigma + (\mu^2 B\tilde{A}^T C + \tilde{A}^{-1})Z^\sigma.$$

Similarly as in the case of system (H), one can prove the Reid roundabout theorem for the system ($\tilde{\mathrm{H}}$) on $[t_0,\infty)$. Condition (D) is then replaced by the eventual strong normality of ($\tilde{\mathrm{H}}$).

Proposition 2.7. *Suppose that* ($\tilde{\mathrm{H}}$) *is eventually strongly normal. Then the following statements are equivalent.*

(i) *System* ($\tilde{\mathrm{H}}$) *is eventually disconjugate, i.e., there exists* $t_0 \in \mathbb{T}$ *such that the solution* (Y,Z) *of* ($\tilde{\mathrm{H}}$) *with* $Y(t_0)=0$ *and* $Z(t_0)=I$*, satisfies* $\operatorname{Ker} Y^\sigma(t) \subseteq \operatorname{Ker} Y(t)$*,* $Y(t)$ *is eventually nonsingular, and for large* t

$$Y^\sigma(t)Y^{-1}(t)\tilde{A}^T(t)C(t) \leq 0$$

(ii) *There exists* $t_0 \in \mathbb{T}$ *such that*

$$\mathcal{F}_\infty(x,u) = \int_{t_0}^{\infty} \left\{(x^\sigma)^T C x^\sigma + u^T B u\right\}(t)\,\Delta t < 0$$

over $\tilde{\mathcal{D}}(t_0)$*, which is defined to be the set of pairs* (x,u) *on* $[t_0,\infty)$ *such that* u *is piecewise rd-continuously delta-differentiable,* x *is piecewise rd-continuous, and satisfying* $u^\Delta(t) = C(t)x^\sigma(t) - A^T(t)u(t)$ *on* $[t_0,\infty)$*,* $u(t_0)=0$*, eventually* $u(t)=0$ *(i.e.,* $u(t)=0$ *for* $t \geq t_1$ *with some* $t_1 > t_0$*), and* $u(t) \not\equiv 0$ *on* $[t_0,t_1]$*.*

(iii) *There exists* $t_0 \in \mathbb{T}$ *and a symmetric matrix* $V(t)$ *satisfying the Riccati equation*

$$(\tilde{\mathrm{R}}) \qquad V^\Delta + B(t) - V^\sigma A^T(t) - [V^\sigma C(t) + A(t)]\tilde{A}(t)[V - \mu(t)B(t)] = 0$$

with $I+\mu(t)C(t)V^\sigma(t)$ *invertible and* $[I+\mu(t)C(t)V^\sigma(t)]^{-1}C(t) \leq 0$ *on* $[t_0,\infty)$*.*

If $(\bar{Y},\bar{Z})$ is the principal solution of the reciprocal system ($\tilde{\mathrm{H}}$) at ∞, then the associated solution $V^+ = ZY^{-1}$ of ($\tilde{\mathrm{R}}$) is eventually maximal in the sense that any solution $V(t)$ of ($\tilde{\mathrm{R}}$) in a neighborhood of ∞ with $I+\mu(t)C(t)V^\sigma(t)$ invertible and $[I+\mu(t)C(t)V^\sigma(t)]^{-1}C(t) \leq 0$ satisfies eventually the inequality $V(t) \leq V^+(t)$.

3. RICCATI INEQUALITY AND DISCONJUGACY

In this section we take $\mathbb{T}=[a,b]$ to be a time scales interval. The first main result of this paper is the following theorem. It relates the disconjugacy of (H) with the solvability of the Riccati dynamic inequality (1.1).

Theorem 3.1 (Riccati inequality). *Suppose that* (D) *holds. Then each of the conditions (i)-(iv) of Proposition 2.4 is also equivalent to*

(v) *There exist delta-differentiable functions* (X, U) *that solve*

$$X^{\Delta} = A(t)X^{\sigma} + B(t)U, \tag{3.1}$$

$$M(t) := (X^{\sigma})^T \left\{U^{\Delta} - C(t)X^{\sigma} + A^T(t)U\right\} \leq 0, \tag{3.2}$$

$t \in [a, b]^{\kappa}$, *with* $X(t)$ *invertible and* $X^T(t)U(t)$ *symmetric for all* $t \in [a, b]$, *and satisfying for all* $t \in [a, b]^{\kappa}$

$$D(t) := X(t)[X^{\sigma}(t)]^{-1}\tilde{A}(t)B(t) \geq 0.$$

(vi) *The Riccati matrix dynamic inequality* (1.1), *i.e.,*

$$\tilde{A}^T(t)\, R[W](t)\, [I + \mu(t)B(t)W]^{-1} \leq 0, \tag{3.3}$$

has a symmetric solution $W(t)$ *on* $[a, b]$ *with* $I + \mu(t)B(t)W(t)$ *invertible and* $[I + \mu(t)B(t)W(t)]^{-1}B(t) \geq 0$ *on* $[a, b]^{\kappa}$.

For the proof we will need the next few results. First note that the Riccati operator $R[W]$ is derived from $W = UX^{-1}$ via the *first* identities of the product rule in Remark 2.1(ii), (iii). However, if we use the *second* identities, then we obtain

$$\begin{aligned} W^{\Delta} = (UX^{-1})^{\Delta} &= U^{\Delta}X^{-1} - U^{\sigma}(X^{\sigma})^{-1}X^{\Delta}X^{-1} \\ &= (C - W^{\sigma}A)\tilde{A}(I + \mu BW) - A^TW - W^{\sigma}BW. \end{aligned}$$

Thus, the Riccati operator and the Riccati equation take the equivalent form

$$R[W](t) = W^{\Delta} + A^T(t)W + W^{\sigma}B(t)W + [W^{\sigma}A(t) - C(t)]\tilde{A}(t)[I + \mu(t)B(t)W] = 0.$$

Note that this equation again reduces to the continuous-time Riccati equation (R_C) when $\mathbb{T} = \mathbb{R}$.

To support the validity of using an inequality in (3.2), and hence in (3.3), we show that the matrix $M(t)$ is indeed symmetric.

Lemma 3.2. *Let* $t \in [a, b]^{\kappa}$ *be fixed. Suppose that* X *and* U *are delta-differentiable functions with* $X^{\Delta} = AX^{\sigma} + BU$ *at* t. *Then the matrix* M *defined by* (3.2) *satisfies*

$$M = (X^TU)^{\Delta} - (X^{\sigma})^TCX^{\sigma} - U^TBU.$$

In particular, if in addition X^TU *is symmetric at* t, *then* M *is symmetric at* t *as well.*

Proof. By using the product rule, we have at t

$$\begin{aligned} M &= (X^{\sigma})^TU^{\Delta} - (X^{\sigma})^TCX^{\sigma} - (AX^{\sigma})^TU \\ &= (X^TU)^{\Delta} - (X^{\Delta})^TU - (X^{\sigma})^TCX^{\sigma} + (X^{\Delta} - BU)^TU \\ &= (X^TU)^{\Delta} - (X^{\sigma})^TCX^{\sigma} - U^TBU, \end{aligned}$$

which shows the claimed identity. The symmetry of M then follows once X^TU is symmetric. □

Lemma 3.3. *Let $t \in [a,b]^\kappa$ be fixed. Suppose X, U, and W are delta-differentiable matrices such that $X^\Delta = AX^\sigma + BU$, X^TU is symmetric, and $WX = UX^\dagger X$ at t. Then at t*

$$(X^\sigma)^T R[W]X = (X^\sigma)^T (U^\Delta - CX^\sigma + A^T U)X^\dagger X = MX^\dagger X. \tag{3.4}$$

Proof. Denote the right-hand side of (3.4) by $\mathbf{R}$. Then

$$\begin{aligned}
\mathbf{R} &= (X^\sigma)^T U^\Delta X^\dagger X - (X^\sigma)^T CX^\sigma X^\dagger X + (X^\sigma)^T A^T WX \\
&= \left\{(X^TU)^\Delta - (X^\Delta)^T U - (X^\sigma)^T CX^\sigma + (X^\sigma)^T A^T WX\right\} X^\dagger X \\
&= \Big\{(X^TWX)^\Delta - (X^\Delta)^T WX - (X^\sigma)^T C\tilde{A}(I+\mu BW)X + (X^\sigma)^T A^T WX\Big\} X^\dagger X \\
&= (X^\sigma)^T \Big\{W^\Delta X + W^\sigma X^\Delta - C\tilde{A}(I+\mu BW)X + A^T WX\Big\} X^\dagger X \\
&= (X^\sigma)^T \Big\{W^\Delta X + (W^\sigma A - C)\tilde{A}(I+\mu BW)X + W^\sigma BWX + A^T WX\Big\} X^\dagger X \\
&= (X^\sigma)^T R[W]XX^\dagger X = (X^\sigma)^T R[W]X,
\end{aligned}$$

where we used the alternative form of the Riccati operator $R[W]$. □

Proof of Theorem 3.1. Conditions (i)-(iv) below refer to Proposition 2.4.

"(iii) ⇒ (v)" This follows trivially, since the solution (X,U) from (iii) satisfies the equality in (3.2).

"(v) ⇒ (i)" Let (X,U) satisfy (v). Then it is a conjoined basis of the linear Hamiltonian system ($\underline{\text{H}}$), where

$$\underline{A} := A, \quad \underline{B} := B, \quad \underline{C} := C + (X^\sigma)^{T-1} M (X^\sigma)^{-1} \le C.$$

Indeed, this is shown by the computation

$$\underline{C}X^\sigma - \underline{A}^T U = CX^\sigma + (X^\sigma)^{T-1} M - A^T U = U^\Delta.$$

Note that $\underline{C}$ is indeed symmetric, by Lemma 3.2. Moreover, (X,U) has no focal points in $(a,b]$. Thus, Proposition 2.3 yields that the quadratic functional $\underline{\mathcal{F}}$ associated with ($\underline{\text{H}}$) is positive definite. Since $\underline{A} = A$ and $\underline{B} = B$, conditions (D) and ($\underline{\text{D}}$) are the same, so that $\underline{\mathcal{F}} > 0$ and ($\underline{\text{D}}$) imply, by Proposition 2.4, that ($\underline{\text{H}}$) is disconjugate on $[a,b]$. Finally, since $\underline{C} \le C$, the Sturm comparison theorem (Corollary 2.6, where the roles of (H) and ($\underline{\text{H}}$) are interchanged) yields that (H) is disconjugate on $[a,b]$ as well.

"(v) ⇒ (vi)" Let (X,U) satisfy (v) and set $W := UX^{-1}$ on $[a,b]$. Then W is symmetric and identity (3.4) shows that

$$(X^\sigma)^T R[W]X = M \le 0.$$

Since $X^\sigma = \tilde{A}(I+\mu BW)X$, the above inequality is equivalent to (3.3). Moreover, $I+\mu BW = \tilde{A}^{-1}X^\sigma X$ is invertible and $(I+\mu BW)^{-1}B = X(X^\sigma)^{-1}\tilde{A}B \geq 0$ on $[a,b]^\kappa$.

"(vi) ⇒ (v)" Suppose that W satisfies (vi). Let $X(t)$ be the solution of the initial value problem

$$X^\Delta = \tilde{A}(t)[A(t)+B(t)W(t)]X, \quad t \in [a,b]^\kappa, \quad X(a) = I.$$

Note that the matrix $\tilde{A}(A+BW)$ is regressive, since $I+\mu\tilde{A}(A+BW) = \tilde{A}(I+\mu BW)$ is invertible. By [18, Theorem 6.2(iii)], $X(t)$ is invertible for all $t \in [a,b]$. If we now set $U := WX$ on $[a,b]$, then X^TU is symmetric and the calculation of X^Δ in the proof of [19, Theorem 2] shows that (3.1) holds. Inequality (3.3) is equivalent to

$$(I+\mu WB)\tilde{A}^T R[W] \leq 0 \quad \Leftrightarrow \quad X^T(I+\mu WB)\tilde{A}^T R[W]X \leq 0,$$

which reduces to $(X^\sigma)^T R[W]X \leq 0$. Thus, by using (3.4), we get

$$M = (X^\sigma)^T R[W]X \leq 0,$$

i.e., (3.2) holds as well. Finally, $X(X^\sigma)^{-1}\tilde{A}B = (I+\mu BW)^{-1}B$ yields that $D \geq 0$ on $[a,b]^\kappa$. □

Remark 3.4. The extension of Theorem 3.1 to *symplectic dynamic systems* (in [3] called *Hamiltonian systems*), which contain linear Hamiltonian systems (H) as a special case [12, 16, 21], is an open problem (even though the Reid roundabout theorem is proven in [21] for these more general linear systems).

4. DISCRETE CASE

In this section we specialize Theorem 3.1 to the discrete case $\mathbb{T} = \mathbb{Z}$. As the theory of linear Hamiltonian difference systems is much more developed than the theory of Hamiltonian systems on time scales, we can easily extend the discrete-time version of Theorem 3.1 to problems with variable endpoints.

Thus, in this section we consider first the system (H_D) and the discrete quadratic functional $\mathcal{F}_D$. The statement of Theorem 3.1 for $\mathbb{T} = \mathbb{Z}$ complements the discrete Reid roundabout theorem of Bohner [8, Theorem 2] and [9, Theorem 4] and reads as follows.

Corollary 4.1. *System* (H) *is disconjugate on* $[0, N+1]$ *iff one of the following conditions hold.*

(i) *There exists matrices* (X, U) *that solve*

$$\Delta X_k = A_k X_{k+1} + B_k U_k, \tag{4.1}$$

$$M_k := X_{k+1}^T \left\{\Delta U_k - C_k X_{k+1} + A_k^T U_k\right\} \leq 0, \tag{4.2}$$

$k \in [0,N]$, *with* X_k *invertible and* $X_k^T U_k$ *symmetric for all* $k \in [0, N+1]$, *and satisfying* $D_k := X_k X_{k+1}^{-1}\tilde{A}_k B_k \geq 0$ *for all* $k \in [0,N]$.

(ii) *The Riccati matrix difference inequality* (1.2)*, i.e.,*

$$\tilde{A}_k^T R_D[W]_k (I + B_k W_k)^{-1} \leq 0, \tag{4.3}$$

has a symmetric solution W_k *on* $[0, N+1]$ *with* $I + B_k W_k$ *invertible and satisfying* $(I + B_k W_k)^{-1} B_k \geq 0$ *for all* $k \in [0, N]$.

Remark 4.2. The statement of Corollary 4.1 applies also to higher order Sturm–Liouville difference equations [7, 10], since the matrix B_k is allowed to be singular.

The positivity of discrete quadratic functionals with separable endpoints was recently studied in [22]. There it was shown that, under a certain $(\mathcal{M}_0 : I)$-normality assumption, the positivity of the quadratic functional

$$\mathcal{G}_D(x, u) := x_0^T \Gamma_0 x_0 + x_{N+1}^T \Gamma\, x_{N+1} + \mathcal{F}_D(x, u)$$

over the endpoint constraints

$$\mathcal{M}_0 x_0 = 0, \quad \mathcal{M} x_{N+1} = 0 \tag{4.4}$$

is equivalent to the solvability of the Riccati difference equation (R_D) with initial equality and final inequality conditions

$$(I - \mathcal{M}_0) W_0 = \Gamma_0, \quad W_{N+1} + \Gamma > 0 \quad \text{on } \operatorname{Ker} \mathcal{M}, \tag{4.5}$$

respectively. Moreover, when no normality is assumed, the initial equality constraint is replaced by a strict inequality while the final inequality remains the same, i.e.,

$$\Gamma_0 - W_0 > 0 \quad \text{on } \operatorname{Ker} \mathcal{M}_0, \quad W_{N+1} + \Gamma > 0 \quad \text{on } \operatorname{Ker} \mathcal{M}. \tag{4.6}$$

Here the $n \times n$-matrices $\mathcal{M}_0$, $\mathcal{M}$ are projections, Γ_0, Γ are symmetric, and without loss of generality $\Gamma_0 = (I - \mathcal{M}_0)\Gamma_0(I - \mathcal{M}_0)$, $\Gamma = (I - \mathcal{M})\Gamma(I - \mathcal{M})$.

Relating the two above mentioned results to the discrete Riccati inequality (4.3) is an obvious question. The parallel continuous-time statements can be found in [27, Theorems 7.2–7.4]. The next two theorems can be shown in a similar matter as Theorem 3.1 and their proofs are therefore omitted.

System (H_D) is called $(\mathcal{M}_0 : I)$-normal on $[0, N+1]$ if the only solution u_k of

$$-\Delta u_k = A_k^T u_k, \quad B_k u_k = 0, \quad k \in [0, N], \quad u_0 = \mathcal{M}_0 \gamma_0,$$

for some vector $\gamma_0 \in \mathbb{R}^n$, is the zero solution $u_k \equiv 0$ on $[0, N+1]$.

Theorem 4.3. *Assume that* (H_D) *is* $(\mathcal{M}_0 : I)$*-normal on* $[0, N+1]$*. Then the following conditions are equivalent to* $\mathcal{G}_D > 0$ *over* (4.4).

(i) *System* (4.1), (4.2) *has a solution* (X, U) *such that* $X_k^T U_k$ *is symmetric and* X_k *is invertible for all* $k \in [0, N+1]$, $D_k \geq 0$ *for all* $k \in [0, N]$*, and satisfying the initial equality and final inequality endpoint constraints*

$$(I - \mathcal{M}_0)(\Gamma_0 X_0 - U_0) = 0, \quad X_{N+1}^T(\Gamma X_{N+1} + U_{N+1}) > 0 \quad \textit{on } \operatorname{Ker} \mathcal{M} X_{N+1}.$$

(ii) *The Riccati matrix difference inequality* (4.3) *has a symmetric solution* W_k *on* $[0, N+1]$ *with* $I + B_k W_k$ *invertible,* $(I + B_k W_k)^{-1} B_k \geq 0$ *for all* $k \in [0, N]$*, and satisfying the initial equality and final inequality endpoint constraints* (4.5).

Theorem 4.4. *The following conditions are equivalent to* $\mathcal{G}_D > 0$ *over* (4.4).

(i) *System* (4.1), (4.2) *has a solution* (X, U) *such that* $X_k^T U_k$ *is symmetric and* X_k *is invertible for all* $k \in [0, N+1]$, $D_k \geq 0$ *for all* $k \in [0, N]$*, and satisfying the initial and final inequality endpoint constraints*

$$X_0^T(\Gamma_0 X_0 - U_0) > 0 \quad on \ \operatorname{Ker} \mathcal{M}_0 X_0, \quad X_{N+1}^T(\Gamma X_{N+1} + U_{N+1}) > 0 \quad on \ \operatorname{Ker} \mathcal{M} X_{N+1}.$$

(ii) *The Riccati matrix difference inequality* (4.3) *has a symmetric solution* W_k *on* $[0, N+1]$ *with* $I + B_k W_k$ *invertible,* $(I + B_k W_k)^{-1} B_k \geq 0$ *for all* $k \in [0, N]$*, and satisfying the initial and final inequality endpoint constraints* (4.6).

Remark 4.5. Discrete Riccati inequalities for quadratic functionals with *jointly* varying endpoints can be obtained by the methods presented in [23]. There it is shown that discrete quadratic functionals with *joint* endpoints can be equivalently studied via quadratic functionals with *separable* endpoints. The method used therein is based on augmenting the given quadratic functional (having jointly varying endpoints) into the double dimension, i.e., with $2n \times 2n$ coefficient matrices, thus obtaining an augmented quadratic functional with *separable* endpoints. Hence, Theorems 4.3, 4.4 can be applied to this transformed problem and the positivity of discrete quadratic functionals with joint endpoints characterized in terms of an (augmented) Riccati inequality with appropriate initial and final endpoint constraints.

5. RECIPROCITY PRINCIPLE

Throughout this section we assume that $\sup \mathbb{T} = \infty$. We start by recalling the fact that the system (H) can always be transformed into a linear Hamiltonian dynamic system of the same form with $A(t) \equiv 0$, i.e.,

$$(\mathrm{H}_R) \qquad x^\Delta = B(t)u, \quad u^\Delta = C(t)x^\sigma,$$

where the oscillatory behavior and normality are preserved. Consequently, we may suppose without loss of generality that $A(t) \equiv 0$ in (H) and ($\tilde{\mathrm{H}}$). Such systems will be called *reduced* linear Hamiltonian dynamic systems. Hence, in this section we deal with the reduced systems (H_R) and

$$(\tilde{\mathrm{H}}_R) \qquad y^\Delta = -C(t)z^\sigma, \quad z^\Delta = -B(t)y.$$

As we have in this case that $\tilde{A}(t) \equiv I$, the associated Riccati equations take the forms

$$(\mathrm{R}_R) \qquad W^\Delta - C(t) + [W^\sigma - \mu(t)C(t)]B(t)W = 0$$

and

$$(\tilde{\mathrm{R}}_R) \qquad V^\Delta + B(t) - V^\sigma C(t)[V - \mu(t)B(t)] = 0,$$

respectively. First we claim that if $[I + \mu(t)B(t)W]^{-1}$ exists, then (R_R) takes the equivalent form

$$W^\Delta - C(t) + W[I + \mu(t)B(t)W]^{-1}B(t)W = 0. \tag{5.1}$$

Indeed, the statement is trivial at the right-dense points, while for t, where $\mu(t) > 0$ (the right-scattered points), equation (R_R) is equivalent to

$$W^\sigma[I + \mu(t)B(t)W] - \mu(t)C(t)[I + \mu(t)B(t)W] - W = 0.$$

This in turn yields

$$\mu(t)W^\Delta - \mu(t)C(t) + W[I + \mu(t)B(t)W]^{-1}[-I + I + \mu(t)B(t)W] = 0,$$

which is equation (5.1) multiplied by a nonzero number $\mu(t)$. Similarly, if $[I + \mu(t)C(t)V^\sigma]^{-1}$ exists, then equation $(\tilde{\mathrm{R}}_R)$ can be rewritten as

$$V^\Delta + B(t) - V^\sigma[I + \mu(t)C(t)V^\sigma]^{-1}C(t)V^\sigma = 0.$$

Following the discrete case and because of the latter form, equation $(\tilde{\mathrm{R}}_R)$ may be called a *time-reversed Riccati equation.*

To prove the main result of this section, which is Theorem 5.4 below, we need the following auxiliary lemmas describing the behavior of certain solutions of the associated Riccati equations.

Lemma 5.1. *Suppose that $C(t) \le 0$ for large t, $W(t)$ is a symmetric solution of (R_R) on $[t_0, \infty)$ such that $I + \mu(t)B(t)W(t)$ is invertible and $[I + \mu(t)B(t)W(t)]^{-1}B(t) \ge 0$ for $t \ge t_0$. If $\bar{W}(t)$ is any symmetric solution of*

$$W^\Delta + W[I + \mu(t)B(t)W]^{-1}B(t)W = 0 \tag{5.2}$$

such that $\bar{W}(t_0) \ge W(t_0)$, then $\bar{W}(t)$ exists on the whole interval $[t_0, \infty)$ and satisfies there the inequalities $[I + \mu(t)B(t)\bar{W}(t)]^{-1}B(t) \ge 0$ and $\bar{W}(t) \ge W(t)$.

Proof. Let (X, U) be the solution of (H_R) given by the initial conditions $X(t_0) = I$, $U(t_0) = W(t_0)$. Then $W(t) = U(t)X^{-1}(t)$, $X(t)$ is nonsingular, and we have $X(t)[X^\sigma(t)]^{-1}B(t) = [I + \mu(t)B(t)W(t)]^{-1}B(t) \ge 0$ for $t \in [t_0, \infty)$. In particular, (X, U) has no focal points in $[t_0, \infty)$. By [21, Theorem 6], this is equivalent to the fact that for any $t_1 > t_0$ we have

$$x^T(t_0)W(t_0)x(t_0) + \int_{t_0}^{t_1} \left\{(x^\sigma)^T C x^\sigma + u^T B u\right\}(t)\,\Delta t > 0, \tag{5.3}$$

where (x, u) satisfies $x^\Delta(t) = B(t)u(t)$ for $t \in [t_0, t_1]^\kappa$, $x(t_1) = 0$, and $x(t) \not\equiv 0$ on $[t_0, t_1]$. Let $(\bar{X}, \bar{U})$ be the solution of $X^\Delta = B(t)U$, $U^\Delta = 0$ given by $\bar{X}(t_0) = I$, $\bar{U}(t_0) = \bar{W}(t_0)$. Then, since $C(t) \le 0$ and $W(t_0) \le \bar{W}(t_0)$, for any $t_1 > t_0$ and for any pair (x, u) satisfying $x^{\Delta}(t) = B(t)u(t)$ on $[t_0, t_1]^{\kappa}$, $x(t_1) = 0$, and $x(t) \not\equiv 0$ on $[t_0, t_1]$, inequality (5.3) implies

$$x^T(t_0)\bar{W}(t_0)x(t_0) + \int_{t_0}^{t_1} \left\{u^T Bu\right\}(t)\,\Delta t > 0.$$

Hence, by [21, Theorem 6], $\bar{X}(t)$ is nonsingular and $\bar{X}(t)[\bar{X}^{\sigma}(t)]^{-1}B(t) = [I + \mu(t)B(t)\bar{W}(t)]^{-1}B(t) \geq 0$ for $t \geq t_0$. Now we prove $\bar{W}(t) \geq W(t)$. Consider the matrix functional

$$\mathcal{F}(X, U) = \int_{t_0}^{t_1} \left\{(X^{\sigma})^T C X^{\sigma} + U^T BU\right\}(t)\,\Delta t.$$

Substituting $(X, U) = (\bar{X}, \bar{U})$ in $\mathcal{F}$ above, using the Picone's identity, see [19], and taking into account the assumption $[I + \mu(t)B(t)W(t)]^{-1}B(t) \geq 0$ we have

$$\begin{aligned}\mathcal{F}(\bar{X}, \bar{U}) &= \left[\bar{X}^T(t)W(t)\bar{X}(t)\right]_{t_0}^{t_1} \\ &\quad + \int_{t_0}^{t_1} \left\{(\bar{U} - W\bar{X})^T(I + \mu BW)^{-1}B(\bar{U} - W\bar{X})\right\}(t)\,\Delta t \\ &\geq \left[\bar{X}^T(t)W(t)\bar{X}(t)\right]_{t_0}^{t_1}.\end{aligned}$$

On the other hand, since $C(t) \leq 0$, we have (again by the Picone's identity)

$$\begin{aligned}\mathcal{F}(\bar{X}, \bar{U}) \leq \int_{t_0}^{t_1} \left\{U^T BU\right\}(t)\,\Delta t &= \left[\bar{X}^T(t)\bar{W}(t)\bar{X}(t)\right]_{t_0}^{t_1} \\ &\quad + \int_{t_0}^{t_1} \left\{(\bar{U} - \bar{W}\bar{X})^T(I + \mu B\bar{W})^{-1}B(\bar{U} - \bar{W}\bar{X})\right\}(t)\,\Delta t \\ &= \left[\bar{X}^T(t)\bar{W}(t)\bar{X}(t)\right]_{t_0}^{t_1}.\end{aligned}$$

Hence,

$$\left[\bar{X}^T(t)W(t)\bar{X}(t)\right]_{t_0}^{t_1} \leq \left[\bar{X}^T(t)\bar{W}(t)\bar{X}(t)\right]_{t_0}^{t_1},$$

and thus we get

$$\bar{X}^T(t_1)\left[W(t_1) - \bar{W}(t_1)\right]\bar{X}(t_1) \leq \bar{X}^T(t_0)\left[W(t_0) - \bar{W}(t_0)\right]\bar{X}(t_0) \leq 0.$$

Since $\bar{X}(t_1)$ is nonsingular, we obtain from the above inequality that $W(t_1) \leq \bar{W}(t_1)$ for any $t_1 > t_0$. □

In the following statements, by $K^{-1}(t) \to 0$ as $t \to \infty$ we mean that $\lambda_{\min}K(t) \to \infty$ as $t \to \infty$, where $\lambda_{\min}$ denotes the smallest eigenvalue of the matrix indicated.

Lemma 5.2. *Suppose that* (H_R) *is eventually disconjugate and eventually identically normal. Let* $W^-(t)$ *be the eventually minimal solution of* (R_R)*. If* $C(t) \leq 0$ *for large* t *and if* $\left(\int^t B(s)\,\Delta s\right)^{-1} \to 0$ *as* $t \to \infty$*, then* $W^-(t) \geq 0$ *for large* t*.*

Proof. Suppose that there exists $\alpha \in \mathbb{R}^n$ and an arbitrarily large $t_0 \in \mathbb{T}$ such that $\alpha^T W^-(t_0)\alpha < 0$. From the equivalent form (5.1) of (R_R) for W^- it follows that $(W^-)^\Delta(t) \le 0$, i.e., the matrix function $W^-(t)$ is nonincreasing. Hence, $\alpha^T W^-(t)\alpha < 0$ for $t \ge t_0$. The assumption $\left(\int^t B(s)\,\Delta s\right)^{-1} \to 0$ as $t \to \infty$ implies that the pair $(X(t), U(t)) \equiv (I, 0)$ is the principal solution of the system $x^\Delta = B(t)u$, $u^\Delta = 0$, by [14, Theorem 3.1]. Thus, $\tilde{W}^-(t) \equiv 0$ is the eventually minimal solution of the corresponding (reduced) Riccati equation (5.2). Let $\bar{W}(t)$ be the solution of (5.2) given by the initial condition $\bar{W}(t_0) = W^-(t_0)$. The assumptions of Lemma 5.1 are satisfied with $W(t) = W^-(t)$ and hence, $\bar{W}(t)$ exists up to ∞ and $\bar{W}(t) \ge W^-(t)$ for $t \ge t_0$. The matrix function $\bar{W}(t)$ is also nonincreasing, thus $\alpha^T \bar{W}(t)\alpha < 0 \equiv \alpha^T \tilde{W}^-(t)\alpha$ for $t \ge t_0$. This however contradicts the fact that $\tilde{W}^-(t) \equiv 0$ is the eventually minimal solution of (5.2). □

By using essentially the same arguments as in Lemmas 5.1 and 5.2, we may prove the following statement.

Lemma 5.3. *Suppose that* ($\tilde{\mathrm{H}}_R$) *is eventually disconjugate and eventually identically normal. Let* $V^+(t)$ *be the eventually maximal solution of* ($\tilde{\mathrm{R}}_R$)*. If* $B(t) \ge 0$ *for large* t *and if* $\left(\int^t C(s)\,\Delta s\right)^{-1} \to 0$ *as* $t \to \infty$*, then* $V^+(t) \le 0$ *for large* t.

The main result of this section is then the following theorem.

Theorem 5.4 (Reciprocity principle)**.** *Suppose that* (H_R) *and its reciprocal system* ($\tilde{\mathrm{H}}_R$) *are eventually identically normal,* $B(t) \ge 0$*, and* $C(t) \le 0$ *for large* t*. Then* (H_R) *is eventually disconjugate iff* ($\tilde{\mathrm{H}}_R$) *is eventually disconjugate.*

Proof. Let (H_R) be eventually disconjugate and $W^-(t)$ be the eventually minimal solution of (R_R). First suppose that $\left(\int^t B(s)\,\Delta s\right)^{-1} \to 0$ as $t \to \infty$, while the other case will be treated later. By Lemma 5.1, $W^-(t) \ge 0$ for large t, say $t \ge t_0$. We will show that actually $W^-(t) > 0$. If not, then there exists $t_1 \ge t_0$ such that $W^-(t_1)$ is singular or negative definite. Since $W^-(t)$ is nonincreasing (use the same argument as at the beginning of the proof of Lemma 5.2), we have that either $W^-(t)$ is singular for $t \in [t_1, \infty)$, which contradicts the eventual identical normality of (H_R), or $W^-(t_2) < 0$ for some $t_2 \ge t_1$. This yields again a contradiction, since $W^-(t) \ge 0$ for $t \ge t_0$. Thus, we have shown that $W^-(t)$ is eventually positive definite. Set $V := -(W^-)^{-1} = -XU^{-1}$. Directly one can verify that $V(t)$ is a solution of ($\tilde{\mathrm{R}}_R$) and, since $W^-(t) > 0$, this solution exists up to infinity. To prove the eventual disconjugacy of ($\tilde{\mathrm{H}}_R$) we need to show that $[I + \mu(t)C(t)V^\sigma(t)]^{-1}C(t) \le 0$ for large t. This holds trivially at right-dense points t. Suppose next that t is right-scattered. Then we have at such t (we omit the argument (t) in the following computations)

$$0 \le -\mu C = U(X^\sigma)^{-1} - U^\sigma(X^\sigma)^{-1} = (X^\sigma)^{T-1}\left[(X^\sigma)^T U - (X^\sigma)^T U^\sigma\right](X^\sigma)^{-1},$$

so that

$$(X^\sigma)^T U \geq (X^\sigma)^T U^\sigma = (X^\sigma)^T (W^-)^\sigma X^\sigma > 0.$$

Thus, $U^{-1}(X^\sigma)^{T-1} \leq (U^\sigma)^{-1}(X^\sigma)^{T-1}$ and

$$\begin{aligned}\mu(I+\mu C V^\sigma)^{-1}C &= \mu\left[(U^\sigma - \mu C X^\sigma)(U^\sigma)^{-1}\right]^{-1} C = \mu\left[U(U^\sigma)^{-1}\right]^{-1} C \\ &= \mu U^\sigma U^{-1} C = U^\sigma U^{-1}\left[U^\sigma (X^\sigma)^{-1} - U(X^\sigma)^{-1}\right] \\ &= U^\sigma\left\{U^{-1}U^\sigma (X^\sigma)^{-1}(U^\sigma)^{T-1} - (X^\sigma)^{-1}(U^\sigma)^{T-1}\right\}(U^\sigma)^T \\ &= U^\sigma\left\{U^{-1}(X^\sigma)^{T-1} - (U^\sigma)^{-1}(X^\sigma)^{T-1}\right\}(U^\sigma)^T \leq 0.\end{aligned}$$

Now, if $\left(\int^t B(s)\,\Delta s\right)^{-1} \not\to 0$ as $t \to \infty$, replace $B(t)$ by a matrix $\bar{B}(t)$ for which $\bar{B}(t) \geq B(t)$ and $\left(\int^t \bar{B}(s)\,\Delta s\right)^{-1} \to 0$ as $t \to \infty$. Then the system $x^\Delta = \bar{B}(t)u$, $u^\Delta = C(t)x^\sigma$ is eventually disconjugate, by the Sturmian comparison theorem. Hence, by Proposition 2.7,

$$\bar{\mathcal{F}}_\infty(x,u) = \int_{t_0}^\infty \left\{(x^\sigma)^T C x^\sigma + u^T \bar{B} u\right\}(t)\,\Delta t < 0$$

for any nontrivial $(x,u) \in \tilde{\mathcal{D}}(t_0)$, t_0 being sufficiently large, where the set $\tilde{\mathcal{D}}(t_0)$ was defined in Proposition 2.7. Since $\bar{B}(t) \geq B(t)$, we have $\mathcal{F}_\infty(x,u) \leq \bar{\mathcal{F}}_\infty(x,u) < 0$. Consequently, $(\tilde{\mathrm{H}}_R)$ is also eventually disconjugate.

Conversely, suppose that $(\tilde{\mathrm{H}}_R)$ is eventually disconjugate and let V^+ be the eventually maximal solution of $(\tilde{\mathrm{R}}_R)$. If $\left(\int^t C(s)\,\Delta s\right)^{-1} \to 0$ as $t \to \infty$, then by Lemma 5.3, $V^+(t) \leq 0$ for large t, and eventual identical normality implies that $V^+(t) < 0$ for large t. Hence $W = -(V^+)^{-1}$ is a solution of (R_R) which exists up to ∞. Similarly as above it may be shown that $[I+\mu(t)B(t)W(t)]^{-1}B(t) \geq 0$ for large t and this means that (H_R) is eventually disconjugate. Finally, if $\left(\int^t C(s)\,\Delta s\right)^{-1} \not\to 0$ as $t \to \infty$, replace $C(t)$ by $\bar{C}(t)$ such that $\bar{C}(t) \leq C(t)$, $\left(\int^t \bar{C}(s)\,\Delta s\right)^{-1} \to 0$ as $t \to \infty$, and use the same argument as in the first part of this proof. □

A more precise formulation of Theorem 5.4 can be obtained from a closer examination of the last proof. Moreover, as we noted at the beginning of this section, systems (H_R) and $(\tilde{\mathrm{H}}_R)$ can be replaced by (H) and $(\tilde{\mathrm{H}})$, respectively. This is a content of the next statement.

Theorem 5.5 (Reciprocity principle). *Suppose that* (H) *and* $(\tilde{\mathrm{H}})$ *are eventually identically normal. If* $C(t) \leq 0$ *for large* t *and if* (H) *is eventually disconjugate, then so is system* $(\tilde{\mathrm{H}})$. *Conversely, if* $B(t) \geq 0$ *for large* t *and if* $(\tilde{\mathrm{H}})$ *is eventually disconjugate, then so is system* (H).

Remark 5.6. The extension of Theorem 5.5 to symplectic dynamic systems [12, 16, 21] is an open problem, since we needed in the proof the fact that the matrix

$I - \mu(t)A(t)$ is invertible. This invertibility assumption is equivalent to the possibility of transforming (H) into the reduced system (H_R), which has $A(t) \equiv 0$.

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